Newform congruences of local origin

Presented by Jackie Voros

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Seemingly random congruence

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Δ is a weight 12 cusp form. That is, it is a weight 12 modular form with no constant term.

$$\Delta(z) \in S_{12}(SL_2(\mathbb{Z}))$$

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n	1	2	3	4	5	6
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And we have that $691 | B_{12}$, so if we consider E_{12} mod 691, the constant term vanishes hence E_{12} is a cusp form mod 691. By the dimension formula, there is only one cusp form of weight 12. So E_{12} and Δ are the same cusp form modulo 691

$$E_{12} \equiv \Delta \pmod{691}$$

as q-expansions.

What does this mean?

- There exists $[a] \in Cl(\mathbb{Q}(\zeta_{691}))[691]$ with

$$\sigma \cdot [a] = \chi_{691}(\sigma)^{-11}[a] \text{ for all } \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$$

  Where $\chi_{691} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to F_{691}^{*}$ is the mod 691 cyclotomic character.

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- Other Eisenstein series congruences prove the Herbrand-Ribet theorem

$$\mathrm{ord}_p(B_k) > 0 \iff \exists \text{ element in } \chi_p^{1-k} \text{ eigenspace of } Cl(\mathbb{Q}(\zeta_p))[p].$$

Next questions

- How do we find the congruences? (Can we find conditions on primes to find congruences?)
- Can this be generalised for higher levels and non-trivial character?
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Yes, all these questions are answered.

Generalisation

Now we look at modular forms of weight k and level N with Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

$$M_k(\Gamma_0(N), \chi)$$

where,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

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And further we consider newforms

- They are 'new' at level N (cannot be mapped to by a lower level modular form)
- Normalised - $f = \sum_{n \geq 0} a_n(f) q^n$, $a_1(f) = 1$
- Eigenforms - eigenvectors for all Hecke operators T_p with $p \nmid N$

Generalisation

Then for Dirichlet characters ψ, φ with $\psi\varphi = \chi$ we have generalised Eisenstein series of weight k and level N, where N is the conductor of χ.

$$E_k^{\psi,\varphi}(z) = \frac{1}{2}\delta(\psi)L(1-k,\psi^{-1}\varphi) + \sum_{n=1}^{\infty}\sigma_{k-1}^{\psi,\varphi}(n)q^n$$

With generalised power divisor series

$$\sigma_{k-1}^{\psi,\varphi}(n) = \sum_{d|n,d>0}\psi(n/d)\varphi(d)d^{k-1}$$

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We say the generlised Eisenstein series is new at level N if

$$\text{conductor of } \psi \times \text{ conductor of } \varphi = N$$

Generalisation

Theorem (??)

Let $E_k^{\psi,\varphi}$ be new at level N with N square-free. Let p be a prime such that $(N,p)=1$ and let $\overline{\chi}$ be the lift of $\chi=\psi\varphi$ to modulus pN.
Let $l>k+1$, $l\nmid 6pN$ be a prime of $\mathbb{Z}[\psi,\varphi]$ and let λ be a prime above l in the ring of integers of the extension of $\mathbb{Q}(\psi,\varphi)$ generated by the Fourier coefficients of f.

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There exists a newform $f\in S_k(\Gamma_0(Np),\overline{\chi})$ with

$$a_q(f)\equiv a_q(E_k^{\psi,\varphi}) \pmod{\lambda}$$

for all primes $q\nmid pNl$ if and only if

- $\text{ord}_l(L(1-k,\psi^{-1}\varphi)(\psi(p)-\varphi(p)p^k))>0$
- $l|(\psi(p)-\varphi(p)^k)(\psi(p)-\varphi(p)p^{k-2})$

Thank you for listening