On the average least negative Hecke eigenvalue

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$f \in H_k^*(N)$ is a primitive, holomorphic Hecke eigenform:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \quad e(z) = e^{2\pi i z},\ \lambda_f(1) = 1$$

That is, for all $n \geq 1$,

$$n^{-(k-1)/2} T_n f = \lambda_f(n) f$$

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Theorem (Kowalski, Lau, Sound., Wu 2011; Matomäki 2012)

The signs of the $\lambda_f(n)$ uniquely determine the eigenform.

- If $\lambda_{f_1}(p)\lambda_{f_2}(p) > 0 \ \forall \ p \in \mathcal{P} \backslash \mathcal{E}$, $\mathcal{E}$ any subset of primes with analytic density $\leq 6/25$, then $f_1 = f_2$.

Motivation

Theorem (Matomäki 2012, following KSIKLSW...)

Let n_f denote the least n such that $\lambda_f(n) < 0$. Then,

$$n_f \ll Q^{3/8}$$

$(Q = k^2 N)$

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- $$\sum_{\substack{n \le x \\ (n,N)=1}} \lambda_f(n) \ll Q^{1/4+\varepsilon} x^{1/2+\varepsilon}$$
- If $\lambda_f(n) > 0$ for $n \le y$, one can construct a contradiction if y is too large.

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Question

$$n_f \overset{?}{\leftrightsquigarrow} n_2(p)$$

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Question

$$n_f \overset{?}{\leftrightsquigarrow} n_2(p)$$

Note that $\lambda_f(n)$ are multiplicative, while $\left(\frac{\cdot}{p}\right)$ is totally multiplicative. Consider p_f, the least prime p such that $\lambda_f(p) < 0$.

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Theorem (Erdős 1961)

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So we may say,

$$\mathbb{P}(n_2(p) = p_i) = \mathbb{P}\left(\left(\frac{p_1}{p}\right) = +1\right) \times \ldots \times \mathbb{P}\left(\left(\frac{p_i}{p}\right) = -1\right) = \frac{1}{2^i}$$

Analogous results

- (Elliott, 1967) Average least k^{th}-power non-residue modulo p
- (Burgess–Elliott, 1968) Average least primitive root modulo p
- (Pollack, 2012) Average least quadratic non-residue modulo m
- (Pollack, 2012) Average least character non-residue
- (Martin–Pollack, 2013) Average least non-split prime in the quadratic field of conductor p
- (Martin–Pollack, 2013) Average least inert prime in a quadratic field
- (Martin–Pollack, 2013) Average least non-split prime over all cubic extensions of $\mathbb{Q}$

Least prime negative Hecke eigenvalue

Analogously, we have,

$$|\lambda_f(n)| \leq \tau(n) \implies \exists\, \theta_f(p) \in [0, \pi] \text{ s.t. } \lambda_f(p) = 2\cos\theta_f(p)$$

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Theorem (Sato-Tate Conjecture) (B-L, G, H, T 2011)

$$\lim_{x\to\infty} \frac{1}{\pi(x)} |\{p \leq x : \theta_f(p) \in [\alpha, \beta], p \nmid N\}| = \int_\alpha^\beta \mathrm{d}\mu_{ST}$$

for $[\alpha, \beta] \subseteq [0, \pi]$, where $\mu_{ST} = \frac{2}{\pi}\sin^2\theta \mathrm{d}\theta$.

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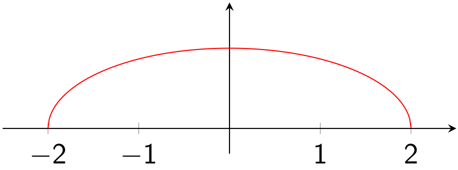

Figure: Distribution of $\lambda_f(p)$ as $p \to \infty$

Average least negative prime Hecke eigenvalue

Theorem (V.)

Assuming GRH we have,

$$\lim_{k+N\to\infty} \frac{1}{|H_k^*(N)|} \sum_{f\in H_k^*(N)} p_f = \sum_{i=1}^{\infty} \frac{p_i}{2^i}.$$

For N square-free, k even.

Why GRH?

Current best unconditional bound on p_f:

Theorem (Thorner 2021)

$$p_f \ll \frac{(kN)^{c(4\log(2e))}}{\log(kN)}$$

The implied constant and c are ‘large’.

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Assuming GRH $L(s, \mathrm{Sym}^m f)$ for each $m \geq 1$ we have,

$$p_f \ll (\log Q)^2.$$

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Multiplicativity through Chebyshev polynomials

$$X_n(\theta_f(p)) = \lambda_f(p^n)$$

where

$$X_n(\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad \theta \in [0, \pi].$$

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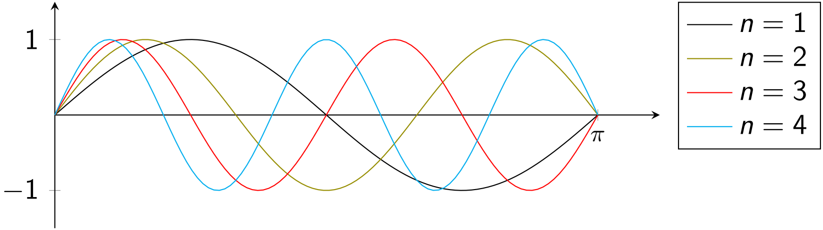
So if $\lambda_f(p^n) > 0$ then $\sin((n+1)\theta_f(p)) > 0$.

Intervals of $\theta_f(p)$

$$\sin((n+1)\theta_f(p)) = \lambda_f(p^n)\sin(\theta_f(p)), \quad \theta_f(p) \in [0,\pi].$$



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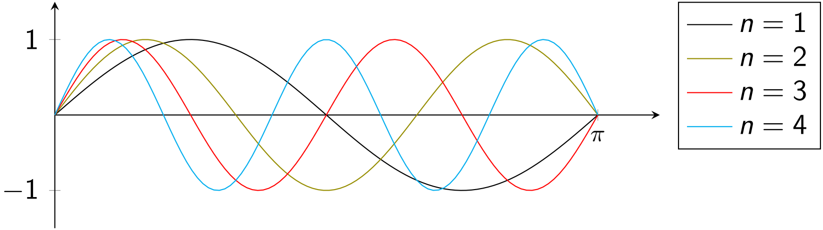

Figure: $\sin((n+1)\theta)$

$\lambda_f(p^n) > 0 \implies \theta_f(p) \in A_n$ where,

$$A_n = \begin{cases} \left(0, \frac{\pi}{n+1}\right) \bigcup \left(\frac{2\pi}{n+1}, \frac{3\pi}{n+1}\right) \bigcup \dots \bigcup \left(\frac{n\pi}{n+1}, \pi\right), n \text{ even} \\ \left(0, \frac{\pi}{n+1}\right) \bigcup \left(\frac{2\pi}{n+1}, \frac{3\pi}{n+1}\right) \bigcup \dots \bigcup \left(\frac{(n-1)\pi}{n+1}, \frac{n\pi}{n+1}\right), n \text{ odd.} \end{cases}$$

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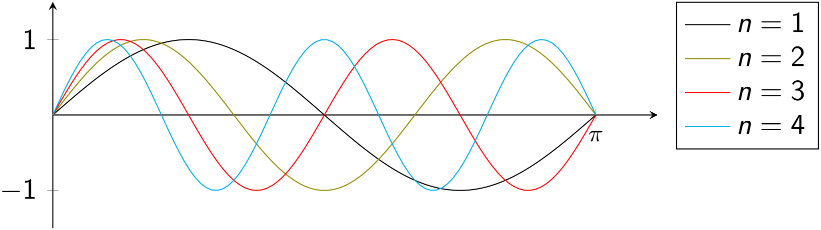

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And $\lambda_f(p^n) < 0 \implies \theta_f(p) \in A_n^c$

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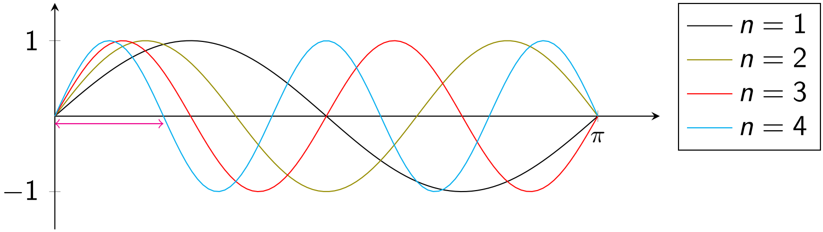

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Then $\lambda_f(p^n) > 0$ for $n = 1, ..., a \implies \theta_f(p) \in \bigcap_{n=1}^{a} A_n = \left(0, \frac{\pi}{a+1}\right)$

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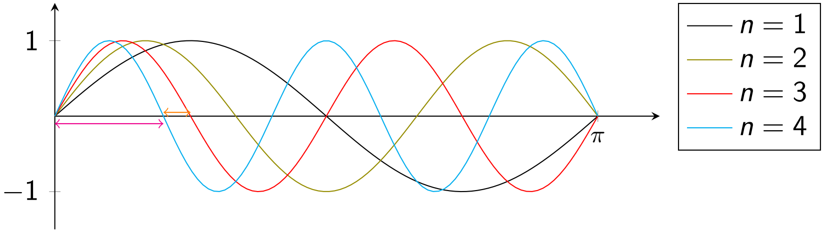

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If $n_f = p^n \implies \lambda_f(p^i) > 0$ for $i < n$ and $\lambda_f(p^n) < 0$

$\implies \theta_f(p) \in \left(\bigcap_{i=1}^{n-1} A_i\right) \bigcap A_n^c = \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$

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Theorem (Serre 1997)

For a fixed prime p we have,

$$\lim_{k+N\to\infty} \frac{1}{|H_k^*(N)|} |\{f \in H_k^*(N) : \theta_f(p) \in [\alpha, \beta]\}| = \int_\alpha^\beta \mathrm{d}\mu_p$$

where $p \nmid N$, $[\alpha, \beta] \subset [0, \pi]$ and where,

$$\mu_p = \frac{2}{\pi}\left(1 + \frac{1}{p}\right) \frac{\sin^2\theta}{(1 - p^{-1})^2 + \frac{4}{p}\sin^2\theta} \mathrm{d}\theta$$

p-adic Plancherel measure

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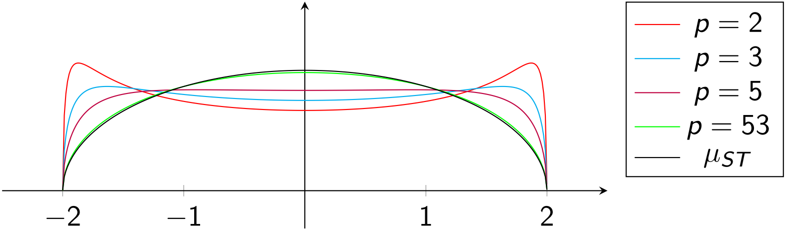

Figure: Distribution of $\lambda_f(p)$ as $k, N \to \infty$

Average n_f

Theorem (V.)

$$\lim_{k+N\to\infty} \frac{1}{|H_k^*(N)|} \sum_{f\in H_k^*(N)} n_f = \sum_{i\geq 1}\sum_{n\geq 1} p_i^n \prod_{j=1}^{\pi(p_i^n)} \mu_{p_j}(I_n(p_j))$$

For N prime or $\log kN$ rough, k even. We have,

$$I_n(p_j) = \begin{cases} \left[0, \frac{\pi}{a_{p_j}(p_i^n)+1}\right] & \text{if } j \neq i, \\ \left[\frac{\pi}{n+1}, \frac{\pi}{n}\right] & \text{if } j = i. \end{cases}$$

And $a_{p_j}(p_i^n) = \lfloor n \log p_i / \log p_j \rfloor$, the greatest power such that $p_j^{a_{p_j}(p_i^n)} < p_i^n$.

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- Atkin-Lehner theory describes the eigenvalues at primes that divide the level.

$$\begin{aligned} \lambda_f(p) &= 0 && \text{if } p^2 | N \\ \lambda_f(p) &= \pm p^{-1/2} && \text{if } p | N \text{ but } p^2 \nmid N \end{aligned}$$

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$$\lambda_f(p) = \pm p^{-1/2} \text{ if } p | N \text{ but } p^2 \nmid N$$

Furthermore, these eigenvalues are totally multiplicative.

$$\lambda_f(p^2) = \lambda_f(p)^2 \quad \text{if } p|N$$

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For a fixed prime p with $p|N$ but $p^2 \nmid N$, and $\omega = \pm 1$,

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This falls from the Eichler-Selberg trace formula. Put simply,

$$TrT_n = A_1 + A_2 + A_3 + A_4$$

For all $n \geq 1$

The trace formula

We can see that for some prime p where $p|N$ and $p^2 \nmid N$, and $\omega = \pm 1$,

$$Tr\left(\frac{\omega p^{-k/2+1} T_p + T_1}{2}\right) = |\{f \in H_k^*(N) : \lambda_f(p) = \omega p^{-1/2}\}|$$

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Via the trace formula, one can show

$$Tr(\omega p^{-k/2+1} T_p) = O(N^{\varepsilon})$$

The trace formula

We can see that for some prime p where $p|N$ and $p^2 \nmid N$, and $\omega = \pm 1$,

$$Tr\left(\frac{\omega p^{-k/2+1}T_p + T_1}{2}\right) = |\{f \in H_k^*(N) : \lambda_f(p) = \omega p^{-1/2}\}|$$

Via the trace formula, one can show

$$Tr(\omega p^{-k/2+1}T_p) = O(N^{\varepsilon})$$

Note that,

$$TrT_1 = |H_k^*(N)|$$

Hence we have

$$\frac{Tr\left(\frac{\omega p^{-k/2-1}T_p + T_1}{2}\right)}{Tr(T_1)} = \frac{1}{2} + O(N^{\varepsilon}).$$

A combined distribution

Theorem

Let $\mathcal{P}$ be a finite set of primes, $(\omega_p)_{p\in\mathcal{P}}$ be a sequence of signs indexed by primes in $\mathcal{P}$. We have,

$$\lim_{k+N\to\infty} \frac{1}{|H_k^*(N)|} |\{f \in H_k^*(N) : \omega_p \lambda_f(p) > 0 \text{ for } p \in \mathcal{P}\}| = \frac{1}{2^{|\mathcal{P}|}}(1 + o(1)),$$

as $k + N$ tends to infinity over even k and square-free N.

Sketch proof

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$$\sum_{f\in H_k^*(N)} p_f = \sum_{p\le A(k,N)} p\cdot \underbrace{\#\{f : p_f = p\}}_{(1+o(1))\frac{1}{2^{\pi(A(k,N))}}|H_k^*(N)|} + \sum_{\substack{f\in H_k^*(N)\\ p_f>A(k,N)}} p_f$$

$$\sum_{f\in H_k^*(N)} n_f = \sum_{n\le B(k,N)} n\cdot \underbrace{\#\{f : n_f = n\}}_{(1+o(1))\prod_{p\le B(k,N)}\mu_p(I_p)|H_k^*(N)|} + \sum_{\substack{f\in H_k^*(N)\\ n_f>B(k,N)}} n_f$$

Provided $A(k,N)$ and $B(k,N)$ tend to infinity slow enough as k and N do.

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This leaves us to show the number of forms with $p_f > A(k, N)$ or $n_f > B(k, N)$ is $o(|H_k^*(N)|)$.

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  2. Stronger bound on a weaker interval

Large sieve inequalities

1. Weaker bound, stronger interval

Let N be square-free, $k \geq 2$ even and let β be a real number. Then there exist two positive constants C_0, C such that,

$$\#\{f \in H_k^*(N) : p_f > \beta\} \ll k\varphi(N) \exp\left(-C\frac{\beta}{\log \beta}\right)$$

Provided k and N are suitably large, and that $C_0 \ll \beta \ll \log(kN)$.

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2. Stronger bound, weaker interval

Let $A > 1$, $0 < \varepsilon_1 < 1$, and $\varepsilon_2 > 0$.

$$\#\{f \in H_k^*(N) : n_f > (\log kN)^A\} \ll k^{1-\varepsilon_1} N^{1/2+\varepsilon_2}$$

Provided k, N large enough.

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$$\sum_{f \in H_k^*(N)} \left| \sum_{P < p \leq Q} b_p \frac{\lambda_f(p^\nu)}{p} \right|^{2j} \leq \sum_{f \in H_k^*(N)} 2j \cdot 2^{2j} \left| \sum_{\substack{P < p \leq Q \\ p \nmid N}} b_p \frac{\lambda_f(p^\nu)}{p} \right|^{2j} + 2j \cdot 2^{2j} \left| \sum_{\substack{P < p \leq Q \\ p \mid N}} b_p \frac{\lambda_f(p^\nu)}{p} \right|^{2j}.$$

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$$\leq \frac{B}{P^{\nu/2} \log P} \to + 2j \cdot 2^{2j} \left| \sum_{\substack{P < p \leq Q \\ p \mid N}} b_p \frac{\lambda_f(p^\nu)}{p} \right|^{2j}.$$

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$$\sum_{f \in H_k^*(N)} \left| \sum_{\beta < p \leq 2\beta} b_p \frac{\lambda_f(p^\nu)}{p} \right|^{2j} \leq \cdots \leq k\varphi(N) \exp\left(-C\frac{\beta}{\log\beta}\right)$$

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In lower bounding, when only considering primes that are coprime to N, we keep N prime or β rough to keep the sets the same.

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For this sieve we use the Deshouillers–Iwaniec sieve for fixed weight,

$$\sum_{f \in H_k^*(N)} \omega_f \Big| \sum_{\substack{m \leq M \\ m \nmid N}} a_m \lambda_f(m) \Big|^2 \ll \left(1 + \frac{M}{N}\right) \|a\|^2, \quad \omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle}$$

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and develop something uniform in both the weight and the level, and use a similar process as the previous sieve.

$$\sum_{f \in H_k^*(N)} \omega_f \Big| \sum_{\substack{m \leq M \\ m \nmid N}} a_m \lambda_f(m) \Big|^2 \ll \left(1 + \frac{M}{Nk^{1-\varepsilon}}\right) \|a\|^2,$$

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We approximate this set through Chebyshev polynomials

$$\prod_{p \le m} Y_p(\theta) = \prod_{p \le m} \sum_{1 \le n_p \le s} \alpha_p(n_p) X_{n_p}(\theta)$$

which form an orthonormal basis of $L^2([0,\pi]^m, \mu_{ST}^{\otimes m})$, and so may approximate the characteristic function of the set.

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$$\prod_{p \le m} Y_p(\theta_f(p)) = \sum_{\substack{d | m^s \\ m | d}} \left(\prod_{p | d} \alpha_p(v_p(m)) \right) \lambda_f(d)$$

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Then apply the hybrid large sieve, picking a_m and M appropriately.

Sketch proof

We can first use the sieve for $\beta = A(k,N)$ or $B(k,N)$ up to $\beta \leq \log(kN)$,

$$\sum_{\substack{f \in H_k^*(N) \\ \beta < (p_f \text{ or } n_f) \leq 2\beta}} \ll \beta \cdot k\varphi(N) \exp\left(-C\frac{\beta}{\log \beta}\right) = o(k\varphi(N)).$$

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Then at $\log kN$ we use the upper bound $n_f \ll Q^{3/8}$ or the GRH bound $p_f \ll (\log Q)^2$, then for large enough k, N (recall $Q = k^2 N$)

$$\sum_{\substack{f \in H_k^*(N) \\ p_f > \log(kN)}} p_f \ll (\log Q)^2 \cdot k\varphi(N) \exp\left(-C\frac{\log kN}{\log\log kN}\right) = o(k\varphi(N))$$

$$\sum_{\substack{f \in H_k^*(N) \\ n_f > (\log(kN))^A}} n_f \ll Q^{3/8} \cdot k^{1-\varepsilon_1} N^{1/2+\varepsilon_2} = o(k\varphi(N))$$

A corollary or two

Clearly, we may make other statements on the signs of $\lambda_f(p)$ depending on how we define N.

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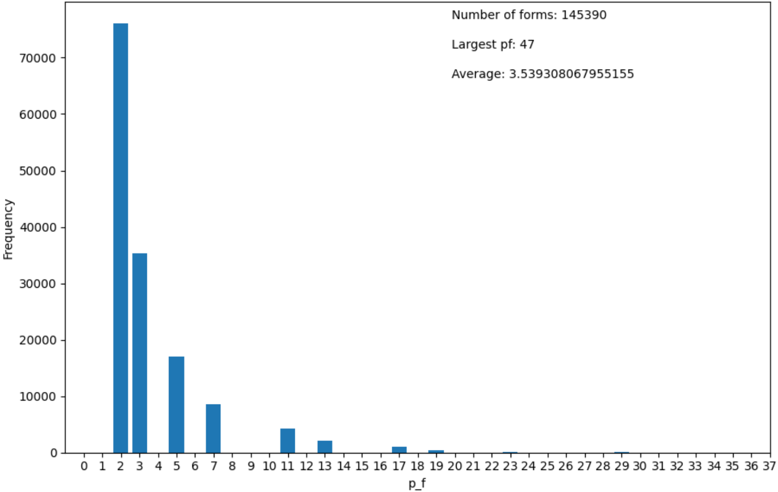
Example

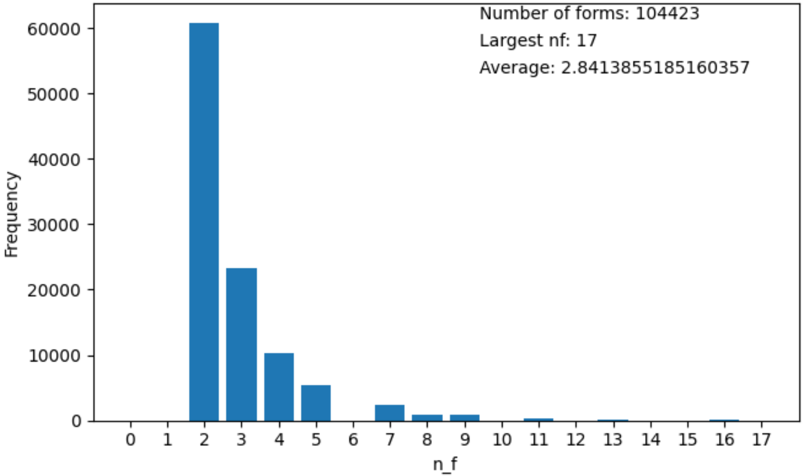
Let $N_a = \prod_{i=1}^{a} p_i$. For k tending to infinity over even integers,

$$\lim_{a,k\to\infty} \frac{1}{|H_k^*(N_a)|} \sum_{f\in H_k^*(N_a)} n_f = \sum_{i=1}^{\infty} \frac{p_i}{2^i},$$

Statistical data - p_f



Statistical data - n_f



Statistical data

$$\sum_{i=1}^{\infty} \frac{p_i}{2^i} \approx 3.674643966011328...$$

$$\sum_{i\geq 1} \sum_{n\geq 1} p_i^n \prod_{j=1}^{\pi(p_i^n)} \mu_{p_j}(I_n(p_j)) \approx 2.9423403000531483...$$

Thank you for listening!

Any questions?