A class of risk processes with reserve-dependent premium rate: sample path large deviations and importance sampling*

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Abstract

Let $(X(t))$ be a risk process with reserve-dependent premium rate, delayed claims and initial capital u. Consider a class of risk processes $\{(X^{\varepsilon}(t)) : \varepsilon > 0\}$ derived from $(X(t))$ via scaling in a slow Markov walk sense, and let $\Psi_{\varepsilon}(u)$ be the corresponding ruin probability. In this paper we prove sample path large deviations for $(X^{\varepsilon}(t))$ as $\varepsilon \to 0$. As a consequence, we give exact asymptotics for $\log \Psi_{\varepsilon}(u)$ and we determine a most likely path leading to ruin. Finally, using importance sampling, we find an asymptotically efficient law for the simulation of $\Psi_{\varepsilon}(u)$.

Keywords: importance sampling; large deviations; Poisson shot noise; risk processes; ruin probabilities.

1 Introduction

Classical risk processes are described by a stochastic process $(X_0(t))$ of the form

$$X_0(t) = u + bt - C(t), \quad t > 0, \tag{1}$$

where u is the initial fortune, $b > 0$ is the gross premium rate, and

$$C(t) = \sum_{n=1}^{N(t)} Z_n$$

is the aggregate claims process. Here $(N(t))$ denotes a Poisson process with intensity $\lambda > 0$; (T_n) are the points of $(N(t))$; (Z_n) is a sequence of independent and identically distributed (i.i.d. for short) nonnegative random variables, independent of $(N(t))$. In order to get more realistic models, classical risk processes were generalised in various directions.

The case where the premium depends on the current reserve was considered by Gerber (1979), Djehiche (1993), and Asmussen and Nielsen (1995). In all these works the authors consider risk processes of the form

$$X_1(t) = u + \int_0^t b(X_1(s))ds - C(t), \quad t > 0, \tag{2}$$

where $b(\cdot)$ is a measurable nonnegative function. Let $\psi^{(1)}(u)$ be the infinite horizon ruin probability corresponding to the model (2) and, for $\varepsilon > 0$, $\psi_{\varepsilon}^{(1)}(u)$ the infinite horizon ruin probability corresponding to (2) with λ/ε and εZ_i in place of λ and Z_i, respectively. Asmussen and Nielsen (1995)

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proved a Lundberg's type inequality for $\psi^{(1)}(u)$, and showed a Cramér-Lundberg type approximation for $\psi_\varepsilon^{(1)}(u)$ in the sense of the slow Markov walk limit (see, for instance, Bucklew (1990)), that is in the limit as $\varepsilon \to 0$. The authors also discuss a fast simulation algorithm for $\psi^{(1)}(u)$, as $u \to \infty$, under a small claim assumption.

Other kinds of generalisations of the classical model (1) account for delay in claim settlement, in order to model the so-called IBNR (Incurred But Not Reported) claims. Works in this direction are due to Waters and Papatriandafylou (1985), Arjas (1989), Neuhaus (1992), and Norberg (1993). More recently, Klüppelberg and Mikosch (1995a) and (1995b) proposed to describe delayed claims in terms of Poisson shot noise processes. More precisely, they consider risk processes of the form

$$X_2(t) = u + bt - S(t), \quad t > 0, \tag{3}$$

where

$$S(t) = \sum_{n=1}^{N(t)} H(t - T_n, Z_n) \tag{4}$$

is a Poisson shot noise process. Here, letting $(E, \mathcal{E})$ denote a measurable space, $H : \mathbb{R} \times E \to [0, \infty[$ is a measurable function such that, for any $z \in E$, $H(\cdot, z)$ is nondecreasing and càdlàg (right-continuous with finite left hand limits) and $H(t, z) = 0$ for all $t \leq 0$; (Z_n) is a sequence of i.i.d. E-valued random variables, independent of $(N(t))$. This model has been studied by Brémaud (2000), who proved a Lundberg's type inequality and a Cramér-Lundberg type approximation for the corresponding infinite horizon ruin probability $\psi^{(2)}(u)$, by Torrisi (2004), who gave a Monte Carlo algorithm for fast simulation of $\psi^{(2)}(u)$ as $u \to \infty$, under a suitable small claim assumption, and by Macci and Torrisi (2004), and Macci, Stabile and Torrisi (2005).

In this paper we combine the ideas underlying the models (2) and (3), considering risk processes which account for reserve-dependent premium rate as well as delay in claim settlement. More precisely, we focus on stochastic processes of the form

$$X(t) = u + \int_0^t b(X(s))ds - S(t), \quad t > 0, \tag{5}$$

where $S(t)$ is given by (4). Let $(X^\varepsilon(t))$ be the strong solution of (5) with λ/ε and $\varepsilon H(t, Z_i)$ in place of λ and $H(t, Z_i)$, respectively (see (6) below); moreover let $\Psi_\varepsilon(u)$ be the corresponding infinite horizon ruin probability. To avoid trivial cases, throughout this work we assume $\mathbb{E}[H(\infty, Z_1)] > 0$. In this paper we use large deviations theory and importance sampling to study the asymptotic properties of $(X^\varepsilon(t))$ and $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$. In particular, we give a Cramér-Lundberg type approximation for $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$, and we find a most likely path leading to ruin; moreover we determine an asymptotically efficient law for the simulation of $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$. These results are based on sample path large deviations of $(X^\varepsilon(t))$ derived by combining the ideas of Freidlin-Wentzell theory (see e.g. Dembo and Zeitouni, 1998, section 5.6, page 212) and the sample path large deviations of $(S(t))$ proved by Ganesh, Macci and Torrisi (2005).

As in the work of Asmussen and Nielsen (1995), the results in this paper on the ruin probability, the most likely path leading to ruin and the asymptotic efficient simulation law are presented in terms of local adjustment coefficients. However we consider different hypotheses; furthermore it is not clear if the techniques of Asmussen and Nielsen can be adapted to risk processes with delayed claims, and we found it more natural to use large deviations theory. The rigourous derivation of the most likely path and the asymptotic efficiency of the simulation law for $\Psi_\varepsilon(u)$ do not appear in the paper of Asmussen and Nielsen.

For the sake of completeness we also discuss the analogies and the differences with the paper of Djehiche (1993). He uses different large deviations techniques to estimate finite horizon ruin probabilities concerning $(X_1(t))$ in (2) in the slow Markov walk sense. The derivation of the most likely path leading to ruin is based on the techniques of calculus of variation, and local adjustment coefficients do not appear.

The paper is structured as follows. In Section 2 we give some preliminaries on large deviations, and we introduce some notations and conditions which will be considered throughout the paper. In Section 3 we prove the sample path large deviations for $(X^\varepsilon(t))$ as $\varepsilon \to 0$. In Section 4 we prove a Cramér-Lundberg type approximation for $\Psi_\varepsilon(u)$, showing an exact asymptotic result for $\log \Psi_\varepsilon(u)$, as $\varepsilon \to 0$. We also determine a most likely path leading to ruin. Finally, in Section 5 we give an asymptotically efficient simulation law for $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$, via importance sampling.

2 Preliminaries

We start recalling some basic definitions in large deviations theory (see, e.g., Dembo and Zeitouni (1998)). A family of probability measures $(\mu_\varepsilon : \varepsilon \in \mathbb{R}_+)$ on a topological space $(M, \mathcal{T}_M)$ satisfies the large deviations principle (LDP for short) with rate function I if $I : M \to [0, \infty]$ is a lower semicontinuous function and the following inequalities hold for every measurable set B:

$$-\inf_{x \in B^\circ} I(x) \le \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(B) \le \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(B) \le -\inf_{x \in \overline{B}} I(x),$$

where B° denotes the interior of B and $\overline{B}$ denotes the closure of B. We also say that a family of M-valued random variables $(Y^\varepsilon : \varepsilon \in \mathbb{R}_+)$ satisfies the LDP if $(\mu_\varepsilon : \varepsilon \in \mathbb{R}_+)$ satisfies the LDP and $\mu_\varepsilon(\cdot) = P(Y^\varepsilon \in \cdot)$. We point out that the lower semicontinuity of I means that its level sets

$$\Phi_I(c) = \{x \in M : I(x) \le c\}, \quad c \ge 0,$$

are closed; when the level sets are compact the rate function I is said to be good.

Throughout this paper we suppose that all the random quantities considered are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with $(\mathcal{F}_t)$ being a complete and right-continuous filtration with respect to which $(S(t))$ is adapted. We notice that $(S(t))$ is a càdlàg process with non-decreasing paths; thus it has finite variation on compact sets. Therefore $(S(t))$ is an $\mathcal{F}_t$-semimartingale (see e.g. Protter, 1990, Theorem 7, page 47) and we can consider the following stochastic differential equation:

$$\begin{cases} dX^\varepsilon(t) = b(X^\varepsilon(t))dt - \varepsilon dS(\frac{t}{\varepsilon}) \\ X^\varepsilon(0) = u \end{cases}, \tag{6}$$

where $\varepsilon, u > 0$ and $b : \mathbb{R} \to [0, \infty[$ is a measurable function. It is known (see e.g. Protter, 1990, Theorem 7, page 197) that there exists a unique strong solution of (6) if b satisfies the following Lipschitz condition:
(L): There exists $L > 0$ such that $|b(x) - b(y)| \le L|x - y|$ for all $x, y \in \mathbb{R}$.

In order to describe the large deviations properties of $S(t)$, we now introduce the following functions:

$$\Lambda_{H(\infty,Z)}(\theta) = \log \mathbb{E}[e^{\theta H(\infty, Z_1)}], \quad \Lambda(\theta) = \lambda(e^{\Lambda_{H(\infty,Z)}(\theta)} - 1), \tag{7}$$

and the Legendre transform of Λ

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}} [\theta x - \Lambda(\theta)]. \tag{8}$$

We point out that $\Lambda'(0) = \lambda \mathbb{E}[H(\infty, Z_1)]$ and $\Lambda^*(\Lambda'(0)) = 0$.

Throughout this paper we always assume the following conditions:
(S): $\Lambda_{H(\infty,Z)}(\theta) < \infty$ for all $\theta \in \mathbb{R}$, and so $\Lambda(\theta) < \infty$ for all $\theta \in \mathbb{R}$ as well.
(A): There exists $B > 0$ such that $\lim_{x \to \infty} b(x) = B$.

(N): $\underline{b} = \inf_{x\in\mathbb{R}} b(x) > \Lambda'(0)$. Hence, $B > \Lambda'(0)$ as well.
Assumption **(S)** is a superexponential condition since it means that the tail of $H(\infty, Z_1)$ goes to zero faster than any exponential rate. Moreover **(A)** gives the asymptotic behaviour of $b(\cdot)$ and **(N)** corresponds to the classical net profit condition.

Observe from (6) and condition **(N)** that $X^\varepsilon(t) \geq \underline{b}t - \varepsilon S(\frac{t}{\varepsilon})$ for all t. Now $\lim_{t\to\infty} S(\frac{t}{\varepsilon}) \ / \ (\frac{t}{\varepsilon}) = \Lambda'(0)$ a.s. (see Klüppelberg and Mikosch, 1995a, Proposition 3.1), so we obtain that

$$\liminf_{t\to\infty} \frac{X^\varepsilon(t)}{t} \geq \underline{b} - \Lambda'(0) = \eta \quad a.s.,$$

for some $\eta > 0$. Since $\lim_{t\to\infty} X^\varepsilon(t) = \infty$ a.s., we also have by **(A)** that

$$\lim_{t\to\infty} \frac{X^\varepsilon(t)}{t} = B - \Lambda'(0).$$

We now introduce some more notation. Let $D[0,\infty[$ be the set of real-valued càdlàg functions defined on $[0,\infty[$. Define for $a, \mu \geq 0$ the sets

$$D_a = \{f \in D[0,\infty[: f(0) = a\}$$

and

$$D_a^\mu = \Big\{ f \in D_a : \lim_{t\to\infty} \frac{f(t)}{1+t} = \mu \Big\}.$$

As in Ganesh, O'Connell and Wischik (2004, page 154) we equip D_a^μ with the metric

$$d_{a,\mu}(f,g) = \sup_{t\geq 0} \frac{|f(t) - g(t)|}{1+t}.$$

Finally we observe that the trajectories of $(\varepsilon S(\frac{t}{\varepsilon}))$ and $(X^\varepsilon(t))$ are elements of $D_0^{\Lambda'(0)}$ and $D_u^{B-\Lambda'(0)}$ respectively, almost surely.

3 Sample path large deviations

In this section we prove a LDP for $(X^\varepsilon(\cdot))$ in $D_u^{B-\Lambda'(0)}$ following the ideas of Freidlin-Wentzell theory. More precisely we obtain the LDP from that of $(\varepsilon S(\frac{\cdot}{\varepsilon}))$ in $D_0^{\Lambda'(0)}$ and the contraction principle (see e.g. Dembo and Zeitouni, 1998, Theorem 4.2.1, page 126). In fact $(\varepsilon S(\frac{\cdot}{\varepsilon}))$ satisfies the LDP in $D_0^{\Lambda'(0)}$ with good rate function

$$I(f) = \begin{cases} \int_0^\infty \Lambda^*(\dot{f}(t))dt & \text{if } f \in AC[0,\infty[\cap D_0^{\Lambda'(0)}, \\ \infty & \text{otherwise,} \end{cases} \tag{9}$$

where $AC[0,\infty[$ is the family of absolutely continuous functions defined on $[0,\infty[$. This follows by the LDP of $(\varepsilon S(\frac{\cdot}{\varepsilon}))$ in $D[0,1]$ equipped with the topology of uniform convergence (see Ganesh, Macci and Torrisi, 2005, Proposition 3.1), using the same techniques in the proof of Theorem 6.2 of Ganesh, O'Connell and Wischik (2004).

Proposition 3.1 *Assume* **(L)**, **(S)**, **(A)** *and* **(N)** *hold. Then* $(X^\varepsilon(\cdot))$ *satisfies the LDP on* $D_u^{B-\Lambda'(0)}$ *with good rate function*

$$J(g) = \begin{cases} \int_0^\infty \Lambda^*(-\dot{g}(t) + b(g(t)))dt & \text{if } g \in AC[0,\infty[\cap D_u^{B-\Lambda'(0)}, \\ \infty & \textit{otherwise.} \end{cases} \tag{10}$$

Proof. As mentioned above $(\varepsilon S(\frac{\cdot}{\varepsilon}))$ satisfies the LDP in $D_0^{\Lambda'(0)}$ with good rate function I in (9). Consider the functional F defined on $D_0^{\Lambda'(0)}$ by $F(f) = g$, where g is the unique càdlàg solution of the integral equation

$$g(t) = u + \int_0^t b(g(s))ds - f(t) \ (t \geq 0). \tag{11}$$

We first show that $F(f) \in D_u^{B-\Lambda'(0)}$. Clearly, $g = F(f)$ is a càdlàg function, and $g(0) = u - f(0) = u$. Choose $\varepsilon > 0$ small enough that $\Lambda'(0) + \varepsilon < \underline{b}$, which is possible by **(N)**. Since $f \in D_0^{\Lambda'(0)}$, there is a $T > 0$ such that

$$\left| \frac{f(t)}{1+t} - \Lambda'(0) \right| \leq \varepsilon$$

for all $t \geq T$. But $g(t) \geq u + \underline{b}t - f(t)$ by (11), and so it follows that $g(t) \to \infty$ as $t \to \infty$. Consequently, by **(A)**, we can choose T so that $|b(g(s)) - B| < \varepsilon$ for all $s > T$. Now, by (11) and the triangle inequality, we have for $t > T$ that

$$\left| \frac{g(t)}{1+t} - [B - \Lambda'(0)] \right| \leq \left| \frac{u + \int_0^T b(g(s))ds - B(1+T)}{1+t} \right| + \left| \frac{\int_T^t [b(g(s)) - B]ds}{1+t} \right| + \left| \frac{f(t)}{1+t} - \Lambda'(0) \right|.$$

The first term on the right goes to zero as $t \to \infty$ since the numerator is a constant. The second and third terms are each bounded by ε as noted above, and so the left hand side is bounded by 3ε for all t large enough. Since $\varepsilon > 0$ is arbitrary, it follows that $g \in D_u^{B-\Lambda'(0)}$.

We shall show later that F is continuous. Since $X^\varepsilon(\cdot) = F(\varepsilon S(\frac{\cdot}{\varepsilon}))$, $(X^\varepsilon(\cdot))$ satisfies the LDP on $D_u^{B-\Lambda'(0)}$ with rate function

$$\widetilde{J}(g) = \inf\{I(f) : F(f) = g\},$$

by the contraction principle. Moreover F is injective, its inverse is

$$[F^{-1}(g)](t) = u + \int_0^t b(g(s))ds - g(t) \ (t \geq 0),$$

and $F^{-1}(g)$ is absolutely continuous if and only if g is absolutely continuous. Therefore $\widetilde{J}(g) = J(g)$, as claimed.

It remains to prove the continuity of F. By **(A)**, there exists $K > 0$ such that $|b(x) - B| < \varepsilon$ for all $x > K$. Fix an arbitrary $f \in D_0^{\Lambda'(0)}$, and an $\varepsilon > 0$ small enough that $\underline{b} - \Lambda'(0) > 2\varepsilon$, which is possible by **(N)**. Then we can find a $T > 0$ such that

$$\left| \frac{f(t)}{1+t} - \Lambda'(0) \right| < \varepsilon \text{ and } u + \underline{b}t - (\Lambda'(0) + 2\varepsilon)(1+t) > K, \tag{12}$$

for all $t > T$. Now set $\delta = \frac{\varepsilon}{(1+T)e^{LT}}$ and let $f_1 \in D_0^{\Lambda'(0)}$ be such that $d_{0,\Lambda'(0)}(f, f_1) < \delta$. Define $g_1 = F(f_1)$ and $g = F(f)$.

Observe that for all $t \geq 0$,

$$\begin{aligned} \frac{|g(t) - g_1(t)|}{1+t} &\leq d_{0,\Lambda'(0)}(f, f_1) + \frac{1}{1+t} \int_0^t |b(g(s)) - b(g_1(s))|ds \\ &\leq \delta + L \int_0^t \frac{|g(s) - g_1(s)|}{1+s} ds, \end{aligned}$$

where L is the Lipschitz constant of $b(\cdot)$. Therefore, by Gronwall's Lemma (see e.g. Elliott, 1982, Lemma 14.20, page 192),

$$\frac{|g(t) - g_1(t)|}{1+t} \leq \delta e^{Lt} \leq \varepsilon \quad \forall t \leq T. \tag{13}$$

From now on let $t > T$ be arbitrarily fixed. Since $d_{0,\Lambda'(0)}(f, f_1) < \delta$ and $\left|\frac{f(t)}{1+t} - \Lambda'(0)\right| < \varepsilon$, it follows that

$$f_1(t) < f(t) + \delta(1+t) < (\Lambda'(0) + \varepsilon + \delta)(1+t) < (\Lambda'(0) + 2\varepsilon)(1+t).$$

Therefore, by (12),

$$g_1(t) = u + \int_0^t b(g_1(s))ds - f_1(t) > u + \underline{b}t - (\Lambda'(0) + 2\varepsilon)(1+t) > K,$$

and likewise, $g(t) > K$. Thus

$$|b(g(t)) - b(g_1(t))| \leq |b(g(t)) - B| + |B - b(g_1(t))| < 2\varepsilon. \tag{14}$$

We also have from (11) that

$$g(t) - g_1(t) = g(T) - g_1(T) + \int_T^t [b(g(s)) - b(g_1(s))]ds - (f(t) - f_1(t)) + (f(T) - f_1(T)).$$

Therefore, by (13), (14) and the assumption that $d_{0,\Lambda'(0)}(f, f_1) < \delta$, we get

$$\frac{|g(t) - g_1(t)|}{1+t} \leq \varepsilon + \frac{2\varepsilon(t-T)}{1+t} + 2\delta < 5\varepsilon.$$

Combining this inequality, which holds for all $t > T$, with (13), which holds for $t \leq T$, yields the continuity of F at f. Since $f \in D_0^{\Lambda'(0)}$ was arbitrary, F is continuous on $D_0^{\Lambda'(0)}$. $\diamondsuit$

4 Ruin probabilities and a most likely path leading to ruin

In this section we provide some large deviations estimates as $\varepsilon \to 0$ for the ruin probabilities

$$\Psi_\varepsilon(u) = P(\exists t \geq 0 : X^\varepsilon(t) \leq 0 | X^\varepsilon(0) = u),$$

and a most likely path leading to ruin.

For $0 \leq v \leq u$, let $w(u, v)$ be defined by

$$w(u, v) = \inf\{J(g) : g \in AC[0, \infty[\cap D_u^{B-\Lambda'(0)} \text{ and } g(t) = v \text{ for some } t \geq 0\}.$$

Proposition 4.1 *Under the assumptions of Proposition 3.1, we have* $\lim_{\varepsilon\to 0} \varepsilon \log \Psi_\varepsilon(u) = -w(u, 0)$ *for any* $u > 0$.

Proof. Consider the function $\mathcal{Q} : D_u^{B-\Lambda'(0)} \to]-\infty, u]$ given by

$$\mathcal{Q}(g) = \inf\{g(t) : t \geq 0\}.$$

Since it is continuous (this can be shown by adapting Theorem 5.3 in Ganesh, O'Connell and Wischik, 2004, page 84), we have by Proposition 3.1 that

$$-J_- \leq \liminf_{\varepsilon\to 0} \varepsilon \log \Psi_\varepsilon(u) \leq \limsup_{\varepsilon\to 0} \varepsilon \log \Psi_\varepsilon(u) \leq -J_+$$

where

$$J_- = \inf\{J(g) : \mathcal{Q}(g) < 0\} \text{ and } J_+ = \inf\{J(g) : \mathcal{Q}(g) \leq 0\}.$$

Since the rate function J is good, the infimum J_+ in the upper bound is attained at some $h \in AC[0, \infty[\cap D_u^{B-\Lambda'(0)}$ such that $\mathcal{Q}(h) \leq 0$. (To see this, note that the set $\{g : \mathcal{Q}(g) \leq 0\}$ is closed as $\mathcal{Q}$ is continuous; we can restrict attention to a compact subset in seeking the infimum of $J(\cdot)$ on this set since $J(\cdot)$ is a good rate function; and finally, since $J(\cdot)$ is lower semicontinuous, it attains its infimum on compact sets.) Since $B - \Lambda'(0) > 0$ by **(N)**, we have $\lim_{t\to\infty} h(t) = \infty$. But $\mathcal{Q}(h) = \inf_{t\geq 0} h(t) \leq 0$, so there must be a $t_0 > 0$ such that $h(t_0) = 0$. Now, if we consider g_h such that

$$g_h(t) = h(t) \text{ for } t \leq t_0 \text{ and } \dot{g}_h(t) = b(g_h(t)) - \Lambda'(0) \text{ for } t > t_0,$$

we have $\mathcal{Q}(g_h) \leq 0$ and $J(g_h) \leq J(h)$. Since h achieves the constrained minimum of J, we can suppose without loss of generality that h satisfies

$$h(t_0) = 0, \quad \text{and} \quad \dot{h}(t) = b(h(t)) - \Lambda'(0),\ t \geq t_0,$$

for some $t_0 > 0$, whence we obtain

$$J_+ = J(h) = \int_0^{t_0} \Lambda^*(-\dot{h}(t) + b(h(t)))dt = w(u,0). \tag{15}$$

We conclude the proof by showing that $J_- = J(h)$. Let $\alpha > 0$ be arbitrarily fixed. Furthermore, for any $\delta > 0$, let g_δ be defined by

$$g_\delta(t) = \begin{cases} h(t) & \text{if } t \leq t_0 \\ -\alpha(t-t_0) & \text{if } t_0 < t \leq t_0+\delta \\ -\alpha\delta + \int_{t_0+\delta}^t b(g_\delta(s))ds - \Lambda'(0)[t-(t_0+\delta)] & \text{if } t > t_0+\delta \end{cases}.$$

Thus $\mathcal{Q}(g_\delta) < 0$ and

$$\begin{aligned} J(g_\delta) &= \int_0^\infty \Lambda^*(-\dot{g}_\delta(t) + b(g_\delta(t)))dt \\ &= \int_0^{t_0} \Lambda^*(-\dot{h}(t) + b(h(t)))dt + \int_{t_0}^{t_0+\delta} \Lambda^*(\alpha + b(-\alpha(t-t_0)))dt + \int_{t_0+\delta}^\infty \Lambda^*(\Lambda'(0))dt \\ &= J(h) + \int_{t_0}^{t_0+\delta} \Lambda^*(\alpha + b(-\alpha(t-t_0)))dt. \end{aligned}$$

(The last equality follows from (15) and $\Lambda^*(\Lambda'(0)) = 0$). Finally,

$$J(h) = J_+ \leq J_- \leq J(h) + \int_{t_0}^{t_0+\delta} \Lambda^*(\alpha + b(-\alpha(t-t_0)))dt$$

and the conclusion follows by letting δ tend to 0, since $\Lambda^*(x)$ is finite for all positive x as $\Lambda'(\theta) \to \infty$ as $\theta \to \infty$, $\Lambda'(\theta) \to 0$ as $\theta \to -\infty$, and $\Lambda'(\cdot)$ is continuous. $\diamondsuit$

An explicit expression for $w(u,0)$ can be derived when the following condition, which corresponds to the classical Cramer's condition, holds:
(C): For all $c \geq \inf_{x\in\mathbb{R}} b(x)$ there exists (a unique) $\gamma_c > 0$ such that $\Lambda(\gamma_c) - c\gamma_c = 0$.
It is easy to check that **(C)** is always satisfied under **(S)** and **(N)**, since $\mathbb{E}[H(\infty, Z_1)] > 0$. Likewise, under the same assumptions, $f'(\gamma_c) = \Lambda'(\gamma_c) - c > 0$. The values $(\gamma_c : c \geq \inf_{x\in\mathbb{R}} b(x))$ are called local adjustment coefficients.

When **(C)** holds, for any $c \geq \inf_{x\in\mathbb{R}} b(x)$ we have the following identity

$$\inf_{t>0} t\Lambda^*\Big(\frac{1}{t} + c\Big) = \frac{\Lambda^*(\Lambda'(\gamma_c))}{\Lambda'(\gamma_c) - c} = \gamma_c, \tag{16}$$

and the infimum in (16) is uniquely attained at $t = \frac{1}{\Lambda'(\gamma_c)-c}$.

Proposition 4.2 *Under the assumptions of Proposition 3.1, we have* $\lim_{\varepsilon\to 0}\varepsilon\log\Psi_\varepsilon(u) = -\int_0^u \gamma_{b(x)}dx$ *for any* $u > 0$.

We start by proving the following lemma.

Lemma 4.3 *For any* $0 \le v \le u$ *we have* $w(u, 0) = w(u, v) + w(v, 0)$.

Proof. Since we trivially have $w(r, r) = 0$ for any $r \ge 0$, we only need to check the case $0 < v < u$. The idea is to check two complementary inequalities.
Inequality 1: $w(u, 0) \le w(u, v) + w(v, 0)$. Let f and g attain the infimum in the definition of $w(u, v)$ and $w(v, 0)$ respectively. The existence of these functions follows from the goodness of $J(\cdot)$, as noted in the proof of Proposition 4.1. Clearly, there exists $t_0 > 0$ such that $f(t_0) = v$ and $s_0 > 0$ such that $g(s_0) = 0$. Defining

$$h(t) = \begin{cases} f(t) & \text{if } t \le t_0, \\ g(t - t_0) & \text{if } t > t_0, \end{cases}$$

we see that h is absolutely continuous and $h(t_0 + s_0) = 0$. Therefore

$$\begin{aligned} w(u, 0) &\le J(h) = \int_0^{t_0} \Lambda^*(-\dot f(t) + b(f(t)))dt + \int_{t_0}^{\infty} \Lambda^*(-\dot g(t - t_0) + b(g(t - t_0)))dt \\ &\le J(f) + J(g) = w(u, v) + w(v, 0). \end{aligned}$$

Inequality 2: $w(u, 0) \ge w(u, v) + w(v, 0)$. Let h attain the infimum in the definition of $w(u, 0)$. Since $\mathfrak{Q}(h) = \inf_{t\ge 0} h(t) \le 0$, there must be a $t_0 > 0$ such that $h(t_0) = 0$. Since $h(0) = u$ and h is continuous, there must be an $s_0 \in (0, t_0)$ such that $h(s_0) = v$. Now let f and g be the continuous functions defined by

$$\begin{cases} f(t) = h(t) & \text{if } 0 \le t < s_0, \\ \dot f(t) = b(f(t)) - \Lambda'(0) & \text{if } t \ge s_0, \end{cases}$$

and

$$\begin{cases} g(t) = h(t + s_0) & \text{if } 0 \le t < t_0 - s_0, \\ \dot g(t) = b(g(t)) - \Lambda'(0) & \text{if } t \ge t_0 - s_0. \end{cases}$$

Then $f(0) = u$, $f(s_0) = v$, $g(0) = v$ and $g(t_0 - s_0) = 0$. Therefore, $w(u, v) \le J(f)$, $w(v, 0) \le J(g)$, and we have

$$\begin{aligned} w(u, 0) &= J(h) \ge \int_0^{t_0} \Lambda^*(-\dot h(t) + b(h(t)))dt \\ &= \int_0^{s_0} \Lambda^*(-\dot f(t) + b(f(t)))dt + \int_{s_0}^{t_0} \Lambda^*(-\dot g(t - s_0) + b(g(t - s_0)))dt \\ &= J(f) + J(g) \ge w(u, v) + w(v, 0). \end{aligned}$$

This completes the proof of the lemma. $\diamondsuit$

Proof of Proposition 4.2. By taking into account Proposition 4.1 and $w(0, 0) = 0$, we only need to check the identity $w'(u, 0) = \gamma_{b(u)}$, where $w'(\cdot, \cdot)$ is the derivative of $w(\cdot, \cdot)$ with respect to the first argument. We have to check right-hand and left-hand derivatives

$$\lim_{\delta\to 0^+} \frac{w(u + \delta, 0) - w(u, 0)}{\delta} \text{ and } \lim_{\delta\to 0^+} \frac{w(u - \delta, 0) - w(u, 0)}{\delta}$$

which can be rewritten as follows by Lemma 4.3:

$$\lim_{\delta\to 0^+} \frac{w(u + \delta, u)}{\delta} \text{ and } \lim_{\delta\to 0^+} -\frac{w(u, u - \delta)}{\delta}.$$

We only consider the right-hand derivative; the left-hand derivative can be checked similarly. Note that

$$w(u+\delta,u)=\inf\{J(g):g\in \mathcal{S}_{u,\delta}\}$$

where

$$\mathcal{S}_{u,\delta}=\{g\in D_{u+\delta}^{B-\Lambda'(0)}:g(t)=u \text{ for some } t\geq 0 \text{ and } \{g(s);s\in[0,t]\}\subset[u,u+\delta]\}\}.$$

Let $g\in AC[0,\infty[\cap \mathcal{S}_{u,\delta}$ be arbitrarily fixed. For t as above, we have

$$\begin{aligned} J(g) &\geq \int_0^t \Lambda^*(-\dot{g}(s)+b(g(s)))ds \geq t\Lambda^*\Big(\frac{1}{t}\int_0^t(-\dot{g}(s)+b(g(s)))ds\Big) & (17)\\ &= t\Lambda^*\Big(\frac{\delta}{t}+\frac{1}{t}\int_0^t b(g(s))ds\Big) \geq \inf_{s\in[u,u+\delta]} t\Lambda^*\Big(\frac{\delta}{t}+b(s)\Big) & (18)\\ &\geq \delta \inf_{s\in[u,u+\delta]}\inf_{t>0}\frac{t}{\delta}\Lambda^*\Big(\frac{1}{t/\delta}+b(s)\Big) = \delta\inf_{s\in[u,u+\delta]}\gamma_{b(s)}, & (19)\end{aligned}$$

where the second inequality in (17) follows from Jensen's inequality since Λ^* is convex, the inequality in (18) holds by the continuity of $b(\cdot)$ and, finally, the equality in (19) follows from (16). Thus

$$w(u+\delta,u)\geq \delta\inf_{s\in[u,u+\delta]}\gamma_{b(s)}. \tag{20}$$

Now let $g\in \mathcal{S}_{u,\delta}\cap AC_0[0,\infty[$ be defined by

$$\begin{cases} g(t)=u-(\Lambda'(\gamma_{b(u)})-b(u))\Big(t-\frac{\delta}{\Lambda'(\gamma_{b(u)})-b(u)}\Big) & \text{if } 0\leq t<\frac{\delta}{\Lambda'(\gamma_{b(u)})-b(u)},\\ \dot{g}(t)=b(g(t))-\Lambda'(0) & \text{if } t\geq \frac{\delta}{\Lambda'(\gamma_{b(u)})-b(u)}. \end{cases}$$

We have

$$\begin{aligned} w(u+\delta,u) &\leq \int_0^{\frac{\delta}{\Lambda'(\gamma_{b(u)})-b(u)}}\Lambda^*(-\dot{g}(t)+b(g(t)))dt\\ &= \int_0^{\frac{\delta}{\Lambda'(\gamma_{b(u)})-b(u)}}\Lambda^*(\Lambda'(\gamma_{b(u)})-b(u)+b(g(t)))dt\\ &\leq \frac{\delta}{\Lambda'(\gamma_{b(u)})-b(u)}\sup_{s\in[u,u+\delta]}\Lambda^*(\Lambda'(\gamma_{b(u)})-b(u)+b(s)) \end{aligned}$$

by taking into account the linearity of g and the continuity of $b(\cdot)$. Thus

$$\limsup_{\delta\to 0}\frac{w(u+\delta,u)}{\delta}\leq \frac{\Lambda^*(\Lambda'(\gamma_{b(u)}))}{\Lambda'(\gamma_{b(u)})-b(u)}=\gamma_{b(u)} \tag{21}$$

by the continuity of Λ^* and of $b(\cdot)$, and by (16).

In conclusion, $\lim_{\delta\to 0^+}\frac{w(u+\delta,u)}{\delta}=\gamma_{b(u)}$ follows by (20), (21) and the continuity of the functions $c\mapsto\gamma_c$ and $b(\cdot)$. ♢

We conclude this section showing that we can find a most likely path leading to ruin, namely a function g which attains the infimum $w(u,0)$ which appears in Proposition 4.1.

Proposition 4.4 *Assume the hypotheses of Proposition 3.1, and let g be the solution of*

$$\dot{g}(t)=-\Lambda'(\gamma_{g(t)})+b(g(t)) \tag{22}$$

with the initial condition $g(0)=u$. Then there exists $t_0>0$ such that $g(t_0)=0$ and

$$\int_0^{t_0}\Lambda^*(-\dot{g}(t)+b(g(t)))dt=\int_0^u\gamma_{b(x)}dx. \tag{23}$$

Proof. Let $c \geq \underline{b}$ be arbitrarily fixed. By **(C)** we have $c = \frac{\Lambda(\gamma_c)}{\gamma_c}$ and therefore

$$\Lambda'(\gamma_c) - c = \lambda e^{\Lambda_{H(\infty,Z)}(\gamma_c)}\Lambda'_{H(\infty,Z)}(\gamma_c) - c = \lambda e^{\Lambda_{H(\infty,Z)}(\gamma_c)}\Lambda'_{H(\infty,Z)}(\gamma_c) - \frac{\Lambda(\gamma_c)}{\gamma_c}.$$

By straightforward computations, we have

$$\begin{aligned} \Lambda'(\gamma_c) - c &= \lambda\Big[e^{\Lambda_{H(\infty,Z)}(\gamma_c)}\Big(\Lambda'_{H(\infty,Z)}(\gamma_c) - \frac{1}{\gamma_c}\Big) + \frac{1}{\gamma_c}\Big] \\ &= \frac{\lambda}{\gamma_c}\mathbb{E}\Big[\gamma_c H(\infty, Z_1)e^{\gamma_c H(\infty,Z_1)} - e^{\gamma_c H(\infty,Z_1)} + 1\Big]. \end{aligned} \tag{24}$$

It is readily verified that the function $f(x) = xe^x - e^x + 1$ is convex and achieves its minimum value zero uniquely at $x = 0$. Hence $\Lambda'(\gamma_c) - c > 0$ for all $\gamma_c > 0$, and thus for all $c \geq \underline{b}$. Moreover, by the assumption that the non-negative random variable $H(\infty, Z_1)$ is not identically zero, the right hand side of (24) goes to infinity as $\gamma_c \to \infty$. Hence, we get

$$\inf_{c \geq \underline{b}} \Lambda'(\gamma_c) - c > 0.$$

It is immediate from this that there exists $t_0 > 0$ such that $g(t_0) = 0$. Now (23) follows by noting that

$$\begin{aligned} \int_0^{t_0} \Lambda^*(-\dot{g}(t) + b(g(t)))dt &= \int_0^{t_0} \Lambda^*(\Lambda'(\gamma_{b(g(t))}))dt \\ &= \int_0^{t_0} \gamma_{b(g(t))} \underbrace{[\Lambda'(\gamma_{b(g(t))}) - b(g(t))]}_{=-\dot{g}(t) \text{ by } (22)} dt = \int_0^u \gamma_{b(x)}dx. \end{aligned}$$

Indeed the first equality holds by (22), the second equality follows by the second equality in (16) with $b(g(t))$ in place of c and the third equality holds by the change of variable $x = g(t)$. $\diamondsuit$

5 Estimation of $\Psi_\varepsilon(u)$ by importance sampling

In this section we address the problem of estimation of $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$, by a Monte Carlo simulation. More precisely, we determine an asymptotically efficient simulation law for $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$. To this end, we use importance sampling (see Glynn and Iglehart (1989), or Bucklew (1990, 2004)).

Throughout this section we assume
(R): E is a Borel subset of $[0, \infty[$; for all $z \in E$ the function $t \mapsto H(t, z)$ is continuous; for all $t \geq 0$ the function $z \mapsto H(t, z)$ is non-decreasing; $H(\infty, z) = z$ for all $z \in E$.

We will need to refer to the process $C(t) = \sum_{n=1}^{N_t} Z_n$, which is assumed to be adapted to $(\mathcal{F}_t)$. For convenience, we also introduce the notation

$$\Lambda_Z(\theta) = \log \mathbb{E}[e^{\theta Z_1}],$$

which is the same as $\Lambda_{H(\infty,Z)}$ by the assumption that $H(\infty, z) = z$ for all $z \in E$. In particular, the function Λ in (7) bears the same relation to Λ_Z as to $\Lambda_{H(\infty,Z)}$. Together with the hypotheses of Proposition 4.2 and **(R)**, we moreover assume the following condition:
(M): The function $b(\cdot)$ is non-decreasing, and $B = \sup_{x\in\mathbb{R}} b(x) < \Lambda'(\gamma_{\underline{b}})$.
The monotonicity of $b(\cdot)$ in **(M)** also appears in Asmussen and Nielsen (1995).

Finally, we notice that

$$\Psi_\varepsilon(u) = P(T_u^\varepsilon < \infty),$$

where

$$T_u^\varepsilon = \inf\{t \geq 0 : X^\varepsilon(t) \leq 0\}$$

is the ruin time of the risk process X^ε, and we point out that the continuity of $t \mapsto H(t,z)$ ensures that T_u^ε is an $(\mathcal{F}_t)$-stopping time. Indeed it is the first exit time from an open set of a process with continuous paths.

5.1 Preliminaries on importance sampling

Consider R independent replications of T_u^ε under the original law P, say $T_{u,1}^\varepsilon, \ldots, T_{u,R}^\varepsilon$. The corresponding crude Monte Carlo estimator of $\Psi_\varepsilon(u)$ is

$$\widehat{\Psi_\varepsilon(u)} = \frac{1}{R}\sum_{i=1}^{R} \mathbf{1}_{T_{u,i}^\varepsilon < \infty},$$

and its relative error is given by

$$\frac{1}{\Psi_\varepsilon(u)}\sqrt{\frac{\Psi_\varepsilon(u)(1-\Psi_\varepsilon(u))}{R}}.$$

Thus, by Proposition 4.2, R needs to grow exponentially with $\frac{1}{\varepsilon}$, as $\varepsilon \to 0$, to keep a fixed relative error. This makes the crude Monte Carlo estimator not efficient to estimate $\Psi_\varepsilon(u)$, as $\varepsilon \to 0$.

To overcome these difficulties, the idea is to consider R independent replications of T_u^ε under another suitable law Q. More precisely, for all $\varepsilon > 0$ and $t \geq 0$, set $\mathcal{F}_t^\varepsilon := \mathcal{F}_{\frac{t}{\varepsilon}}$, and define another law Q on $(\Omega, \mathcal{F})$ in such a way that Q is absolutely continuous with respect to P on $\mathcal{F}_t^\varepsilon$, with a strictly positive density $\frac{dQ_t^\varepsilon}{dP_t^\varepsilon}$, and Q is admissible for simulations, namely $Q(T_u^\varepsilon < \infty) = 1$ (for all $\varepsilon > 0$). The importance sampling estimator of $\Psi_\varepsilon(u)$ is by definition

$$[\widehat{\Psi_\varepsilon(u)}]_Q = \frac{1}{R}\sum_{i=1}^{R}\frac{dP_{T_u^\varepsilon}^\varepsilon}{dQ_{T_u^\varepsilon}^\varepsilon}\mathbf{1}_{T_{u,i}^\varepsilon < \infty}.$$

Note that $[\widehat{\Psi_\varepsilon(u)}]_Q$ is an unbiased estimator of $\Psi_\varepsilon(u)$ under Q, and its variance is given by

$$\begin{aligned}\mathrm{Var}_Q[[\widehat{\Psi_\varepsilon(u)}]_Q] &= \frac{\mathbb{E}_Q\Big[\Big(\frac{dP_{T_u^\varepsilon}^\varepsilon}{dQ_{T_u^\varepsilon}^\varepsilon}\Big)^2\mathbf{1}_{T_u^\varepsilon<\infty}\Big] - (\Psi_\varepsilon(u))^2}{R} =^{\text{by } Q(T_u^\varepsilon<\infty)=1} \\ &= \frac{\mathbb{E}_Q\Big[\Big(\frac{dP_{T_u^\varepsilon}^\varepsilon}{dQ_{T_u^\varepsilon}^\varepsilon}\Big)^2\Big] - (\Psi_\varepsilon(u))^2}{R}.\end{aligned}$$

To get an asymptotically efficient simulation law we have to minimize the variance of the importance sampling estimator in some sense. The second moment

$$\eta_Q(\varepsilon) = \mathbb{E}_Q\Big[\Big(\frac{dP_{T_u^\varepsilon}^\varepsilon}{dQ_{T_u^\varepsilon}^\varepsilon}\Big)^2\Big]$$

is the only term of the variance which depends on Q, and, for a fixed ε, its minimization is often intractable. Following Siegmund's criterion (see Siegmund, 1976; see also Lehtonen and Nyrhinen, 1992), we say that an admissible law Q is asymptotically efficient, as $\varepsilon \to 0$, if

$$\lim_{\varepsilon\to 0}\varepsilon\log\eta_Q(\varepsilon) = -2\int_0^u \gamma_{b(x)}dx. \tag{25}$$

Note that if the number of replications R has to be chosen to guarantee a fixed relative error of $[\widehat{\Psi_\varepsilon(u)}]_Q$, then R has a chance of growing at less than an exponential rate in $1/\varepsilon$ if and only if (25) holds.

5.2 An asymptotically efficient simulation law

Let $\varepsilon > 0$ be arbitrarily fixed, and Y^ε the strong solution of

$$\begin{cases} dY^\varepsilon(t) = b(Y^\varepsilon(t))dt - \varepsilon dC(\frac{t}{\varepsilon}) \\ Y^\varepsilon(0) = u \end{cases}.$$

Moreover set

$$\Lambda_{\varepsilon Z}(\theta) = \Lambda_Z(\varepsilon\theta) \text{ and } \Lambda_\varepsilon(\theta) = \frac{\lambda}{\varepsilon}(e^{\Lambda_{\varepsilon Z}(\theta)} - 1);$$

if **(C)** holds, for all $c \geq \inf_{x\in\mathbb{R}} b(x)$ there exists (a unique) $\gamma_c^\varepsilon > 0$ such that $\Lambda_\varepsilon(\gamma_c^\varepsilon) - c\gamma_c^\varepsilon = 0$ and $\Lambda'_\varepsilon(\gamma_c^\varepsilon) - c > 0$. Then, since $\Lambda_\varepsilon(\theta) - c\theta = \frac{\Lambda(\varepsilon\theta)-c\varepsilon\theta}{\varepsilon}$, we get the equalities

$$\varepsilon\gamma_c^\varepsilon = \gamma_c \text{ and } \Lambda_{\varepsilon Z}(\gamma_c^\varepsilon) = \Lambda_Z(\gamma_c). \tag{26}$$

Let $\widetilde{P}$ be another law on $(\Omega, \mathcal{F})$ such that, for all $\varepsilon > 0$ and $t \geq 0$, $\widetilde{P}$ is absolutely continuous with respect to P on $\mathcal{F}_t^\varepsilon$ with density

$$\Big(\frac{d\widetilde{P}_t^\varepsilon}{dP_t^\varepsilon}\Big) = \exp\Big(-\int_0^t \gamma^\varepsilon_{b(Y^\varepsilon(s-))} dY^\varepsilon(s)\Big) =^{\text{by (26)}} \exp\Big(-\frac{1}{\varepsilon}\int_0^t \gamma_{b(Y^\varepsilon(s-))} dY^\varepsilon(s)\Big);$$

it is commonly referred to as an exponential tilting or twisting of the original probability law.

As pointed out by Asmussen and Nielsen (1995, Proposition 3), $\widetilde{P}$ makes Y^ε a risk process with arrival rate $\lambda_x^{(\varepsilon)}$ and claim size distribution $\widetilde{P}_{\varepsilon Z}^{(x)}$ depending on the current level $Y^\varepsilon(t) = x$, and given by

$$\lambda_x^{(\varepsilon)} = \frac{\lambda}{\varepsilon} e^{\Lambda_{\varepsilon Z}(\gamma^\varepsilon_{b(x)})} =^{\text{by (26)}} \frac{\lambda}{\varepsilon} e^{\Lambda_Z(\gamma_{b(x)})}$$

and

$$d\widetilde{P}_{\varepsilon Z}^{(x)}(y) = \frac{e^{\gamma^\varepsilon_{b(x)} y}}{e^{\Lambda_{\varepsilon Z}(\gamma^\varepsilon_{b(x)})}} dP_{\varepsilon Z}(y) =^{\text{by (26)}} \frac{e^{\frac{\gamma_{b(x)}}{\varepsilon} y}}{e^{\Lambda_Z(\gamma_{b(x)})}} dP_{\varepsilon Z}(y). \tag{27}$$

Here $P_{\varepsilon Z}$ is the common law of the random variables (εZ_n) under the original probability law.

In the proof of Proposition 5.1 below, we denote the usual stochastic order by $\leq_{st}$, and the likelihood ratio order by $\leq_{lr}$; we refer the reader to Müller and Stoyan (2002, page 2 and page 12) for the definitions.

Proposition 5.1 *Assume* **(M)**, **(R)** *and the hypotheses of Proposition 3.1. Then $\widetilde{P}$ is an admissible law, i.e. $\widetilde{P}(T_u^\varepsilon < \infty) = 1$ $(\forall\varepsilon > 0)$.*

Proof. Throughout this proof we simply write $Y(t)$ in place of $Y^1(t)$. We notice that, under $\widetilde{P}$, $(N(t))$ is a Cox process with stochastic intensity $(\lambda e^{\Lambda_Z(\gamma_{b(Y(t))})})$, and we denote the points of $(N(t))$ by (T_n). We have $\lambda e^{\Lambda_Z(\gamma_{b(Y(t))})} \geq \lambda e^{\Lambda_Z(\gamma_{\underline{b}})}$ for all $t \geq 0$, $\widetilde{P}$ almost surely. Then (see e.g. Müller and Stoyan, 2002, pp. 211-216) we can define a homogeneous Poisson process $(\overline{N}(t))$ with intensity $\lambda e^{\Lambda_Z(\gamma_{\underline{b}})}$ on the same probability space of $(N(t))$, with points $(\overline{T}_n)$, such that $(\overline{T}_n) \subset (T_n)$. Therefore

$$S(t) = \sum_{n=1}^{N(t)} H(t - T_n, Z_n) \geq \sum_{n=1}^{\overline{N}(t)} H(t - \overline{T}_n, Z_n),\ \widetilde{P} \text{ a.s.}. \tag{28}$$

On the same probability space we can also define a sequence of i.i.d. random variables $(\overline{Z}_n)$, independent of $(\overline{N}(t))$, with law $\widetilde{P}_{\overline{Z}}$ having density $\frac{e^{\gamma_{\underline{b}} y}}{e^{\Lambda_Z(\gamma_{\underline{b}})}}$ with respect to the common law P_Z of the random variables (Z_n) under P.

Let $\widetilde{P}_{Z_n}^{(x)}$ be the law of Z_n given $\{Y(T_n) = x\}$. By (27), $\widetilde{P}_{Z_n}^{(x)}$ has density $\frac{e^{\gamma_{b(x)} y}}{e^{\Lambda_Z(\gamma_{b(x)})}}$ with respect to P_Z; then $\overline{Z}_n \leq_{lr} [Z_n|\{Y(T_n) = x\}]$ and therefore $\overline{Z}_n \leq_{st} Z_n$ (see e.g. Müller and Stoyan, 2002, Theorems 1.4.5 and 1.3.8). Thus, by a well-known result on the usual stochastic order (see e.g. Müller and Stoyan, 2002, Theorem 1.2.4) we can think that

$$\overline{Z}_n \leq Z_n \quad \widetilde{P} \text{ a.s..} \tag{29}$$

Thus, by (28), **(R)** and (29) we have

$$S(t) = \sum_{n=1}^{N_t} H(t - T_n, Z_n) \geq \sum_{n=1}^{\overline{N}_t} H(t - \overline{T}_n, \overline{Z}_n) := \overline{S}(t), \ \widetilde{P} \ a.s.,$$

and therefore

$$\liminf_{t\to\infty} \frac{S(t)}{t} \geq \liminf_{t\to\infty} \frac{\overline{S}(t)}{t} = \Lambda'(\gamma_{\underline{b}}), \ \widetilde{P} \ a.s.$$

(for the a.s. limit see Klüppelberg and Mikosch, 1995a; see also Torrisi, 2004). The conclusion follows noticing that, for any $\varepsilon > 0$, the following inequalities hold $\widetilde{P}$ almost surely:

$$\limsup_{t\to\infty} \frac{X^\varepsilon(t)}{t} \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t b(X^\varepsilon(s))ds - \liminf_{t\to\infty} \frac{S(t/\varepsilon)}{t/\varepsilon} \leq B - \Lambda'(\gamma_{\underline{b}}) < 0,$$

where the latter inequality is guaranteed by **(M)**. $\diamondsuit$

Proposition 5.2 *Assume* **(M)**, **(R)** *and the hypotheses of Proposition 3.1. Then* $\widetilde{P}$ *is an asymptotically efficient simulation law, i.e. it is admissible and*

$$\lim_{\varepsilon\to 0} \varepsilon \log \eta_{\widetilde{P}}(\varepsilon) = -2\int_0^u \gamma_{b(x)}dx. \tag{30}$$

Proof. The admissibility of $\widetilde{P}$ has been proved in the previous Proposition 5.1. We now show (30) proving suitable lower and upper bounds.
Lower bound. Note that

$$\liminf_{\varepsilon\to 0} \varepsilon \log \eta_{\widetilde{P}}(\varepsilon) = \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{\widetilde{P}}\Big[\Big(\frac{dP^\varepsilon_{T^\varepsilon_u}}{d\widetilde{P}^\varepsilon_{T^\varepsilon_u}}\Big)^2 \mathbf{1}_{T^\varepsilon_u<\infty}\Big] \geq$$

$$\geq 2\liminf_{\varepsilon\to 0} \varepsilon \log \underbrace{\mathbb{E}_{\widetilde{P}}\Big[\Big(\frac{dP^\varepsilon_{T^\varepsilon_u}}{d\widetilde{P}^\varepsilon_{T^\varepsilon_u}}\Big) \mathbf{1}_{T^\varepsilon_u<\infty}\Big]}_{=\Psi_\varepsilon(u)} = -2\int_0^u \gamma_{b(x)}dx.$$

Indeed the first equality holds by Proposition 5.1, the inequality holds by Jensen's inequality and the latter equality holds by Proposition 4.2.
Upper bound. For the upper bound we need two preliminary results:

$$Y^\varepsilon(t) \leq X^\varepsilon(t) \ (\forall t \geq 0), \text{ almost surely,} \tag{31}$$

and

$$\int_0^{T^\varepsilon_u} \gamma_{b(Y^\varepsilon(s-))}dY^\varepsilon(s) \leq -f(u), \quad \text{where } f(u) := \int_0^u \gamma_{b(x)}dx. \tag{32}$$

We first show (31). By the definition of Y^ε and X^ε and by taking into account the inequality $S(t) \leq C(t)$ (for all $t \geq 0$) we get

$$Y^\varepsilon(t) - X^\varepsilon(t) = \int_0^t [b(Y^\varepsilon(s)) - b(X^\varepsilon(s))]ds - \varepsilon\Big(C\Big(\frac{t}{\varepsilon}\Big) - S\Big(\frac{t}{\varepsilon}\Big)\Big) \leq \int_0^t [b(Y^\varepsilon(s)) - b(X^\varepsilon(s))]ds =$$

$$= \int_0^t [b(Y^\varepsilon(s)) - b(X^\varepsilon(s))]\mathbf{1}_{Y^\varepsilon(s)\le X^\varepsilon(s)} ds + \int_0^t [b(Y^\varepsilon(s)) - b(X^\varepsilon(s))]\mathbf{1}_{Y^\varepsilon(s)> X^\varepsilon(s)} ds.$$

Since $b(\cdot)$ is nondecreasing we have

$$Y^\varepsilon(t) - X^\varepsilon(t) \le \int_0^t [b(Y^\varepsilon(s)) - b(X^\varepsilon(s))]\mathbf{1}_{Y^\varepsilon(s)> X^\varepsilon(s)} ds.$$

By **(L)**, we obtain

$$Y^\varepsilon(t) - X^\varepsilon(t) \le L\int_0^t (Y^\varepsilon(s) - X^\varepsilon(s))\mathbf{1}_{Y^\varepsilon(s)> X^\varepsilon(s)} ds = L\int_0^t (Y^\varepsilon(s) - X^\varepsilon(s))^+ ds$$

and therefore

$$(Y^\varepsilon(t) - X^\varepsilon(t))^+ \le L\int_0^t (Y^\varepsilon(s) - X^\varepsilon(s))^+ ds \ (\forall t \ge 0).$$

For any fixed ω, the function $s \mapsto (Y^\varepsilon(s,\omega) - X^\varepsilon(s,\omega))^+$ is locally bounded since it is a càdlàg function. Then, by Gronwall's Lemma, we have $(Y^\varepsilon(t) - X^\varepsilon(t))^+ \le 0$ for all $t \ge 0$, almost surely, and (31) follows.

We now show (32). By Theorem 31 of Protter (1990, page 71) with $f(t) = \int_0^t \gamma_{b(x)} dx$ and $V_t = Y^\varepsilon(t)$ we have

$$\int_0^t \gamma_{b(Y^\varepsilon(s-))} dY^\varepsilon(s) = f(Y^\varepsilon(t)) - f(Y^\varepsilon(0)) - \sum_{0<s\le t}\Big\{f(Y^\varepsilon(s)) - f(Y^\varepsilon(s-)) - f'(Y^\varepsilon(s))(Y^\varepsilon(s) - Y^\varepsilon(s-))\Big\} =$$

$$= f(Y^\varepsilon(t)) - f(Y^\varepsilon(0)) - \sum_{0<s\le t}\int_{Y^\varepsilon(s-)}^{Y^\varepsilon(s)} \{f'(x) - f'(Y^\varepsilon(s-))\}dx.$$

Then, by setting $t = T_u^\varepsilon$, we have

$$\int_0^{T_u^\varepsilon} \gamma_{b(Y^\varepsilon(s-))} dY^\varepsilon(s) = f(Y^\varepsilon(T_u^\varepsilon)) - f(Y^\varepsilon(0)) - \sum_{0<s\le T_u^\varepsilon}\int_{Y^\varepsilon(s-)}^{Y^\varepsilon(s)} \{f'(x) - f'(Y^\varepsilon(s-))\}dx =$$

$$= f(Y^\varepsilon(T_u^\varepsilon)) - f(u) + \sum_{0<s\le T_u^\varepsilon}\int_{Y^\varepsilon(s)}^{Y^\varepsilon(s-)} \{\gamma_{b(x)} - \gamma_{b(Y^\varepsilon(s-))}\}dx.$$

We point out that $X^\varepsilon(T_u^\varepsilon) \le 0$ by definition, and therefore $Y^\varepsilon(T_u^\varepsilon) \le 0$ by (31). Since $Y^\varepsilon(t) \le Y^\varepsilon(t-)$ for all $t > 0$, and the functions $b \mapsto \gamma_b$ and $b(\cdot)$ are nondecreasing, we have

$$\sum_{0<s\le T_u^\varepsilon}\int_{Y^\varepsilon(s)}^{Y^\varepsilon(s-)} \{\gamma_{b(x)} - \gamma_{b(Y^\varepsilon(s-))}\}dx \le 0,$$

and (32) follows.

Proof of the upper bound. By taking into account the definition of the function f, the upper bound can be checked as follows:

$$\limsup_{\varepsilon\to 0} \varepsilon \log \eta_{\widetilde{P}}(\varepsilon) = \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{\widetilde{P}}\Big[\exp\Big(\frac{2}{\varepsilon}\int_0^{T_u^\varepsilon} \gamma_{b(Y^\varepsilon(s-))} dY^\varepsilon(s)\Big)\Big] \le^{\text{by } (32)}$$

$$\le \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{\widetilde{P}}\Big[\exp\Big(-\frac{2}{\varepsilon} f(u)\Big)\Big] = -2f(u) = -2\int_0^u \gamma_{b(x)} dx. \ \diamondsuit$$

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