Rumours, consensus and epidemics on networks: Exercises

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1. Let T_1 and T_2 be exponential random variables with parameters λ_1 and λ_2 respectively, and let $T = \min\{T_1, T_2\}$.

   (a) Show that the distribution of T is $\text{Exp}(\lambda_1 + \lambda_2)$.

   (b) Show that the probability that $T = T_1$ is $\lambda_1/(\lambda_1 + \lambda_2)$, and that this is independent of the value of T. (*Hint.* Compute $P(T = T_1 | T = t)$ and show that this doesn't depend on t. Hence infer the unconditional probability that $T = T_1$.)

2. (a) We say that a random variable N has a Geometric distribution with parameter p, written $N \sim Geom(p)$ if

   $$P(N = k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

   Let $N \sim Geom(p)$, and let $T_1, T_2, T_3, \ldots$ be iid $\text{Exp}(\lambda)$ random variables, independent of N. Let $T = \sum_{k=1}^{N} T_N$. Using moment generating functions or otherwise, show that T is exponentially distributed with parameter λp. (*Hint.* Recall that the moment generating function of T is defined as $M(\theta) = \mathbb{E}[\exp(\theta T)]$. First compute $\mathbb{E}[\exp(\theta T) | N = n]$ and then average over N to obtain the unconditional expectation.)

   (b) Let $X_t, t \geq 0$ be a Poisson process of rate λ_1, and let $Y_1, Y_2, Y_3, \ldots$ be iid Bernoulli(p) random variables. Recall that this means that $Y_i = 1$ with probability p and $Y_i = 0$ with probability $1 - p$.

   Let $X_t^1 = \sum_{i=1}^{X_t} Y_i$ be the process obtained by retaining each point of the Poisson process X_t independently with probability p and discarding it with probability $1-p$. It is called the Bernoulli(p) thinning of the Poisson process X_t.

Show that $X^1_t, t \geq 0$ is a Poisson process of rate λp by showing that the times between successive events are $\mathrm{Exp}(\lambda p)$, and any other properties required.

3. Let S_n be the star graph on n nodes consisting of a single hub node connected to each of $n-1$ leaves; there are no edges between leaves. Consider the following rumour-spreading model. Nodes become active according to independent unit rate Poisson processes. If the hub becomes active, it chooses a leaf uniformly at random and informs it of the rumour if it is already informed. If a leaf becomes active, it informs the hub if it is already informed.

   (a) Suppose that only the hub node knows the rumour at time 0. Compute $\mathbb{E}[T_{k+1} - T_k]$ exactly, and use this to compute $\mathbb{E}[T_n]$ exactly.

   (b) Compute the conductance$\Phi(P)$ for the star graph with the probabilities specified above, and the corresponding upper bound on $\mathbb{E}[T_n]$, and compare it with the exact answer.

   (c) Repeat the exact analysis when only a single leaf node initially knows the rumour.

4. Let C_n be the cycle graph on n nodes numbered $\{1, 2, 3, \ldots, n\}$, where there are two directed edges out of each node i. These go to nodes $i-1$ and $i+1$ for $2 \leq i \leq n-1$. The edges out of node 1 go to nodes 2 and n, while the edges out of node n go to nodes $n-1$ and 1. Consider the rumour-spreading model in which nodes become active according to independent unit rate Poisson processes, and an active node contacts one of its two neighbours chosen uniformly at random and informs it of the rumour if it knows the rumour.

   (a) Suppose that only a single node knows the rumour at time 0. Compute $\mathbb{E}[T_{k+1} - T_k]$ exactly, and use this to compute $\mathbb{E}[T_n]$ exactly. (*Hint.* Observe that, at any time, the set of nodes that knows the rumour has to be a contiguous set.)

   (b) Compute the conductance $\Phi(P)$ for the cycle graph with the probabilities specified above, and the corresponding upper bound on $\mathbb{E}[T_n]$, and compare it with the exact answer.

5. Let $G = (V, E)$ be a *directed* graph on n nodes. Consider the following rumour-spreading model on G. There are n independent Poisson processes, $\{N_v(t), t \geq 0\}$, one associated with each node $v \in V$. The Poisson process at node v has rate λ_v. If there is an increment of the process $N_v(\cdot)$ at time t, then node v chooses ones of its neighbours w at random, with probability p_{vw}, which is the vw^{th} element of a stochastic matrix P. If node v knows the rumour at time t, then node w learns it at this time; if neither or both nodes know the rumour, there is no change.

   Let T_k be the first time that exactly k nodes know the rumour, and suppose that $T_1 = 0$, i.e., a single node knows the rumour to start with.

   (a) The above 'node-driven' model is equivalent to the following 'edge-driven' model. There are independent Poisson processes on the edges, with r_{ij} denoting the rate of the process on edge (i, j). Whenever there is an increment of the Poisson process on edge (i, j), node i informs node j of the rumour if i already knows it and j does not. If both or neither of i and j know the rumour, there is no change.

   Explicitly compute the value of r_{ij} for each (i, j), in terms of the rates λ_v and matrix P given above, so that the equivalence holds. You don't have to prove the equivalence.

   (b) For the model in part (a), let R denote the matrix with elements r_{ij}, and define its generalised conductance

   $$\Psi(R) = \min_{S \subset V : S \neq \emptyset} \frac{\sum_{i \in S, j \in S^c} r_{ij}}{\frac{1}{n}|S| \cdot |S^c|},$$

   where S^c denotes the complement of the subset S.

   Show that for each k between 1 and $n-1$, $T_{k+1} - T_k$ is stochastically dominated by an $Exp\big(\frac{1}{n}k(n-k)\Psi(R)\big)$ random variable. You may use the fact that if $\alpha \geq \beta$, then an $Exp(\alpha)$ random variable is stochastically dominated by an $Exp(\beta)$ random variable.

   (c) Use the answer to part (b) to obtain an upper bound on $\mathbb{E}[T_n]$, the time until all nodes learn the rumour. Specifically, show that

   $$\mathbb{E}[T_n] \leq \frac{2(1 + \log n)}{\Psi(R)}.$$

You may use the fact that

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq 1 + \log(n).$$

(All logarithms are natural.)

6. Consider the complete undirected graph $G = (V, E)$. Suppose each edge (v, w) has a random length drawn from an Exp(1) distribution, and that the lengths of different edges are mutually independent. Fix a node $s \in V$. For any other node $v \in V$, the distance from s to v, denoted $d(s, v)$, is defined as the minimum of the lengths of all paths between s and v. The length of a path is the sum of the lengths of the edges constituting the path. The distance $d(s, s)$ is defined to be zero. Finally, let $D_s = \max_{v \in V} d_{s,v}$ denote the maximum distance from s to any another node in the graph.

   Compute the mean of the random variable D_s (or a good bound on it) by reducing the problem to one you know how to solve. Explain your reasoning carefully.

7. Let S_n denote the star graph, which consists of a hub connected to each of $n - 1$ leaves; there are no edges between leaves. Consider the following voter model on S_n. Each node becomes active at the points of a Poisson process of rate 1, independent of all other nodes. When it becomes active, it chooses a neighbour uniformly at random from the set of all its neighbours (i.e., excluding itself), and copies the state of that neighbour.

   Denote by $X_v(t) \in \{0, 1\}$ the state of node v at time t. Let $M(t) = (n - 1)X_{\text{hub}}(t) + \sum_{v \neq \text{hub}} X_v(t)$.

   (a) Show that $M(t)$ is a martingale.

   (b) Suppose that initially the hub and $k - 1$ leaves are in state 1, while $n - k$ leaves are in state 0. What is the probability of being absorbed into the all-1 state?

8. Consider the following modification of the classical voter model on the complete graph K_n. Nodes can be in one of two states, 0 or 1, and

change state as follows. Each node v becomes active at the points of a Poisson process of rate λ, independent of all other nodes. It then contacts a node w chosen uniformly at random from among all n nodes (including itself). If w has the same state as v, nothing happens. Otherwise, v copies the state of w with probability p, independent of everything in the past; with the residual probability $1-p$, it retains its current state. (You can think of this as modelling an attachment to one's current opinion / preference /affiliation.)

Suppose that initially, at time zero, k nodes are in state 1 and $n-k$ nodes are in state 0. Let T denote the random time that the process hits one of the absorbing states, either the all-zero state, denoted $\mathbf{0}$, or the all-one state, denoted $\mathbf{1}$.

(a) Compute the probability of hitting the all-one state.

(b) Obtain an upper bound on the expectation of T, the random time to absorption.

Hint. You may, if you wish, use the results derived in lectures for the above model with $p=1$. These results state that $\mathbb{P}_k(\text{hit } \mathbf{1}) = k/n$, and $\mathbb{E}_k[T] \leq n/\lambda$.

9. Let C_n be the cycle graph on n nodes, namely the graph in which n nodes are arranged around a circle, and each is connected to its nearest neighbour on the right and left. All edges are undirected. Consider the SIS epidemic, or contact process, on C_n, with infection rate α and cure rate 1.

   (a) Let N_t denote the number of infected nodes at time t, and S_t the set of infected nodes. Explain why

   $$\mathbb{E}[N_{t+dt} - N_t | S_t] \leq (2\alpha - 1)N_t dt + o(dt),$$

   for any set S_t of infected nodes, of size N_t.

   (b) Use the answer to the last part to obtain an upper bound on $\mathbb{E}[N_t]$ for arbitrary $t \geq 0$ and arbitrary initial conditions.

   You may use without proof the fact that, if $x'(t) \leq \alpha x(t)$ for all t, where $x'(t)$ denotes the derivative of x at t, then $x(t)$ is no bigger than the solution of the differential equation $y'(t) = \alpha y(t)$ started with the same *positive* initial condition $y(0) = x(0)$.

(c) Use the answer to the last part to obtain an upper bound on the expectation of the random time $T = \inf\{t \geq 0 : N_t = 0\}$, the time for the epidemic to die out.