A class of risk processes with delayed claims: ruin probabilities estimates under heavy-tailed conditions

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Abstract

We consider a class of risk processes with delayed claims, and we provide ruin probabilities estimates under heavy-tailed conditions on the claim size distribution.

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1 Introduction

In this paper we are concerned with stochastic processes with drift of the form

$$S(t) = \sum_{n \geq 1} F(t - T_n) Z_n \mathbf{1}_{(0,t]}(T_n) - ct, \tag{1}$$

where $\{T_n\}_{n \geq 1}$, $T_0 = 0$, are the points (epochs) of a renewal process, whose inter-arrival times $U_n = T_n - T_{n-1}$ have finite mean, $\{Z_n\}_{n \geq 1}$ is a sequence of independent and identically distributed (iid) positive random variables (rv's), independent of $\{T_n\}_{n \geq 1}$, $c > 0$ is a positive constant, and $F(\cdot)$ is a distribution function (df) such that $F(t) = 0$ for $t < 0$.

An important example of the stochastic model considered above arises in insurance risk theory; see Klüppelberg and Mikosch (1995a, 1995b), Mikosch and Nagaev (1998), Brémaud (2000) and Klüppelberg, Mikosch and Schärf (2003), where Poisson shot noise processes are considered. In Brémaud (2000), large deviations theory is used to give Cramér-Lundberg type estimates of the infinite horizon ruin probability. Other models of delayed claims (not based on shot noise processes) are considered in Waters and Papatriandafylou (1985), and more recently in Yuen, Guo and Ng Kai (2005), where a martingale approach is used to estimate the infinite horizon ruin probability under light-tailed conditions.

The interpretation of the process $\{S(t)\}_{t \geq 0}$ in the insurance context is the following. Suppose that claims $\{Z_n\}_{n \geq 1}$ occur according to a renewal process $\{T_n\}_{n \geq 1}$, and the insurance company honors the claim Z_n, which occurred at time T_n, at the rate $f(\cdot - T_n) Z_n$, with $f(\cdot)$ being a probability density on $(0, \infty)$. Then the total claim paid in the time interval $(0, t]$ is

$$\sum_{n \geq 1} F(t - T_n) Z_n \mathbf{1}_{(0,t]}(T_n),$$

where $F(\cdot)$ is the df with density $f(\cdot)$. If the insurance company receives premium income at constant rate $c > 0$, then $S(t)$ defined above is the excess of claims over premiums. If we assume that the

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insurance company has an initial capital $u > 0$, then we say that ruin occurs at the first time t that $S(t) \geq u$, if there is such a time. This leads us to define the infinite horizon ruin probability

$$\psi(u) = P\left(\sup_{t \geq 0} S(t) \geq u\right) \tag{2}$$

and the finite horizon ruin probability

$$\psi(u, T) = P\left(\sup_{t \in [0,T]} S(t) \geq u\right), \tag{3}$$

where $T > 0$ is a positive constant.

In this paper, under heavy-tailed assumptions on the distribution of Z_1, we give asymptotic estimates for $\psi(u)$ and $\psi(u, e(u)T)$ as $u \to \infty$, where $e(u) = \mathrm{E}[Z_1 - u \mid Z_1 > u]$ is the mean excess function of Z_1. Our results exploit the heavy-tail intuition which predicts exceedances of level $u > 0$ to occur as the consequence of one big jump. In particular, we show that the classical ruin probability estimate (see Teugels and Veraverbeke (1973), Veraverbeke (1977), and Embrechts and Veraverbeke (1982)) holds unchanged for $\psi(u)$. Likewise, assuming Poisson claim arrivals, we show that some well-known estimates of the finite horizon ruin probability of the Cramér-Lundberg model (see Asmussen and Klüppelberg (1996)) hold also for $\psi(u, e(u)T)$. This is an insensitivity property of the model considered, in that the asymptotic behavior of $\psi(u)$ and $\psi(u, e(u)T)$ depends only on the distribution of Z_1, not on the shape or nature of the shot. It is an analogue of an insensitivity property in the light-tailed case with Poisson claim arrivals (see Brémaud (2000) for the infinite horizon case, and Macci, Stabile and Torrisi (2005) for the finite horizon case) where the large deviation rate functions of $\psi(u)$ and $\psi(u, uT)$ do not depend on the shape of the shot. We also give a short proof of the insensitivity property in the light-tailed case for completeness.

Our technique is based on a recent work by Asmussen and Albrecher (2006), where risk processes with shot noise Cox claim arrivals are considered. A closely related work is Asmussen, Schmidli and Schmidt (1999) where the heavy-tailed behavior of the infinite horizon ruin probability of risk processes with ergodic or regenerative input is studied.

In the literature on shot noise models, one usually deals with shot shapes of the form $h(t, z)$ (in place of the multiplicative form $F(t)z$) where $h(\cdot, z)$ is a non-decreasing function for each z. The extension of the results in this paper to this more general situation is an open problem.

The paper is structured as follows. We recall some preliminaries and introduce some notation in Section 2. Our results on ruin probabilities with heavy-tailed claim sizes are given in Section 3. Finally, we provide ruin probability estimates for light-tailed claim sizes in the Appendix; while these estimates coincide with those in Brémaud (2000), the derivation is simpler.

2 Preliminaries

Recall that a df $G(\cdot)$ is said subexponential if its support is $(0, \infty)$ and $\overline{G}^{*2} \sim 2\overline{G}$ (see, for instance, Rolski et al. (1999)). Here $\overline{G} = 1 - G$ denotes the tail of the df G, $\overline{G}^{*2}(\cdot)$ denotes the two-fold convolution of $\overline{G}(\cdot)$, and we write $g_1 \sim g_2$ if the functions $g_1(\cdot), g_2(\cdot)$ are such that $\lim_{x\to\infty} g_1(x)/g_2(x) = 1$; we write $g_1(x) = o(g_2(x))$ if $\lim_{x\to\infty} g_1(x)/g_2(x) = 0$. We say that a positive function $g(\cdot)$ on $(0, \infty)$ is regularly varying at infinity of index $\alpha \in \mathbb{R}$, and we write $g \in \mathcal{R}(\alpha)$, if $g(x) \sim x^\alpha L(x)$ as $x \to \infty$, where $L(\cdot)$ is a slowly varying function, that is $\lim_{x\to\infty} L(tx)/L(x) = 1$ for each $t > 0$.

The family of subexponential df's will be denoted by $\mathcal{S}$. It can be further classified using extreme value theory. Goldie and Resnick (1988) showed that if $G \in \mathcal{S}$ and satisfies some smoothness conditions, then G belongs to the maximum domain of attraction of either the Frechet distribution

$\Phi_\alpha(x) = \mathrm{e}^{-x^{-\alpha}}$ or the Gumbel distribution $\Lambda(x) = \mathrm{e}^{-\mathrm{e}^{-x}}$. Moreover, in the former case, it has regularly varying tail of index $-\alpha$.

Throughout this paper we denote by $B(\cdot)$ the df of Z_1 and by $B_0(\cdot)$ its integrated tail df:

$$\overline{B_0}(u) = \frac{1}{\mu}\int_u^\infty \overline{B}(x)\,\mathrm{d}x \quad u > 0 \quad \text{where } \mu = E[Z_1].$$

We assume that $B_0 \in \mathcal{S}$ and that either $\overline{B} \in \mathcal{R}(-\alpha - 1)$, or that B belongs to the maximum domain of attraction of the Gumbel, written $B \in MDA(\Lambda)$. We also assume the classical net profit condition:

$$\rho = \mu/(c\nu) < 1, \quad \text{where } \nu = E[U_1].$$

This condition says that the mean rate c at which premium income is earned exceeds the mean rate μ/ν at which claims need to be paid out. If this condition doesn't hold, then ruin is certain.

3 Ruin probabilities

In this section we derive asymptotic estimates for $\psi(u)$ and $\psi(u, e(u)T)$ as $u \to \infty$, under heavy-tailed conditions on Z_1. To this end, we compare the process $\{S(t)\}_{t\geq 0}$ defined in (1) with the process $\{C(t)\}_{t\geq 0}$ given by

$$C(t) = \sum_{n\geq 1} Z_n \mathbf{1}_{(0,t]}(T_n) - ct. \tag{4}$$

Clearly, the following domination holds:

$$S(t) \leq C(t) \quad \text{a.s., for all } t \geq 0. \tag{5}$$

We shall use this to obtain upper bounds on the ruin probabilities in both the infinite and finite horizon settings. Lower bounds will be obtained by comparing $S(\cdot)$ with a different classical risk process.

3.1 The infinite horizon case

Let $\Psi(\cdot)$ denote the infinite horizon ruin probability for the risk process $\{C(t)\}_{t\geq 0}$. By (5), $\psi(u) \leq \Psi(u)$ for all $u > 0$. Therefore, by the classical ruin estimate (see Teugels and Veraverbeke (1973), Veraverbeke (1977) and Embrechts and Veraverbeke (1982); see also Theorem 6.5.11 in Rolski et al. (1999)), we have for arbitrary $B_0 \in \mathcal{S}$ that

$$\limsup_{u\to\infty} \frac{\psi(u)}{\overline{B_0}(u)} \leq \lim_{u\to\infty} \frac{\Psi(u)}{\overline{B_0}(u)} = \frac{\rho}{1-\rho}. \tag{6}$$

We shall obtain matching lower bounds, adapting to our context the techniques in Albrecher and Asmussen (2006). We begin by bounding the risk process $S(\cdot)$ from below. For all $t \geq 0$ and $a > 0$, define the risk processes:

$$\check{S}_a(t) = \sum_{n\geq 1} F(a) Z_n \mathbf{1}_{(0,t]}(T_n) - ct \tag{7}$$

and

$$\check{C}_a(t) = \sum_{n\geq 1} Z_n \mathbf{1}_{(0,t]}(T_n) - (c/F(a))t \tag{8}$$

It is clear using the monotonicity of $F(\cdot)$ that

$$S(t) \geq \check{S}_a(t-a) - ca \quad \text{a.s., for all } t > a. \tag{9}$$

Therefore, letting $\check{\Psi}_a(\cdot)$ denote the infinite horizon ruin probability relative to the risk process $\{\check{C}_a(t)\}_{t\geq 0}$ we get:

$$\psi(u) \geq \check{\Psi}_a((u+ca)/F(a)) \quad \text{for all } u > 0. \tag{10}$$

We shall use this to obtain asymptotic lower bounds on the ruin probability matching the upper bound in (6) under additional conditions on the claim size distribution.
Our first result states that, if the claim sizes have regularly varying tails, then the ruin probability with delayed claims is insensitive to the shape of the shot, in the sense that it is asymptotically equivalent to the ruin probability in the classical model. More precisely:

Proposition 3.1 *If $\overline{B} \in \mathcal{R}(-\alpha-1)$ for some $\alpha > 0$, then*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{B_0}(u)} = \frac{\rho}{1-\rho}. \tag{11}$$

If the claim size distribution has lighter tails, then we need additional assumptions on the shape of the shot in order to retain asymptotic equivalence with the classical model. The first part of our next result shows that if the shot has compact support, then (11) holds so long as the claim size distribution has an integrated tail which is subexponential. The second part shows that if the tail of the shot decays sufficiently rapidly relative to the mean excess function of the claim size distribution, then (11) continues to hold.

Proposition 3.2

(a) If $B_0 \in \mathcal{S}$ and $F(\cdot)$ has compact support, then (11) *holds.*

(b) Suppose $B \in MDA(\Lambda)$, $B_0 \in \mathcal{S}$ and that $e(u) \sim g(u)$ as $u \to \infty$, for some eventually non-decreasing function $g(\cdot)$. Suppose further that there is a $\gamma > 0$ such that

$$u\overline{F}(u^{1/\gamma}) = o(e(u)) \tag{12}$$

and

$$u^{1/\gamma} = o(e(u)) \tag{13}$$

as $u \to \infty$. Then (11) *holds.*

Before proving these propositions, we work out some examples of the conditions imposed on the shot shape, under part (*b*) of Proposition 3.2, by some heavy-tailed distributions of practical interest. These examples show that the assumptions of part (*b*) are not too restrictive.

Examples

(a) *Weibull distribution*. Suppose the claim size distribution has tail $\overline{B}(u) = \mathrm{e}^{-u^\alpha}$ for $u \geq 0$, where $\alpha \in (0,1)$ is the shape parameter of the Weibull distribution. It is well-known that $B \in MDA(\Lambda)$ and $B_0 \in \mathcal{S}$ (see, for instance, Embrechts et al. (1997)). Moreover, by partial integration we get:

$$e(u) \sim \frac{u^{1-\alpha}}{\alpha} \quad \text{as } u \to \infty,$$

and therefore $e(u)$ is asymptotically equivalent to a non-decreasing function. Now (12) can be rewritten as $\overline{F}(u) = o(e(u^\gamma)/u^\gamma) = o(u^{-\gamma\alpha})$. Hence (12) and (13) hold provided there is a $\gamma > 1/(1-\alpha)$ such that $\overline{F}(u) = o(u^{-\gamma\alpha})$. In particular, this is the case if F is either light-tailed or has regularly varying tail of index $-\beta$ with $\beta > \alpha/(1-\alpha)$.

(b) *Lognormal distribution*. Denote by $\Phi(\cdot)$ the df of the standard normal distribution and take $B(u) = \Phi((\ln u - \omega)/\sigma)$, $u > 0$, where $\omega \in \mathbb{R}$ and $\sigma > 0$ are given constants. It is well-known that $B \in MDA(\Lambda)$ and $B_0 \in \mathcal{S}$ (see, for instance, Asmussen and Klüppelberg (1996)). Furthermore, using Mill's ratio and l'Hospital's rule, we get

$$e(u) \sim \frac{\sigma^2 u}{\ln u - \omega} \quad \text{as } u \to \infty.$$

Hence $e(u)$ is asymptotic equivalent to an eventually non-decreasing function, and (13) is satisfied for any $\gamma > 1$. Also, (12) is satisfied if $\overline{F}(u^{1/\gamma}) = o(1/\ln u)$, i.e., $\overline{F}(u) = o(1/\ln u)$, irrespective of γ. In other words, the conditions of Proposition 3.2(b) are satisfied provided the shot has tail decaying faster than logarithmically.

(c) *Benktander distributions*. We first consider the so-called Benktander distribution of type I (see, for instance, Embrechts et al. (1997)):

$$\overline{B}(u) = (1 + 2(\delta/\alpha)\ln u)\exp\{-(\delta(\ln u)^2 + (\alpha+1)\ln u)\} \quad u \geq 1,\ \alpha, \delta > 0.$$

We have that $B \in MDA(\Lambda)$, $B_0 \in \mathcal{S}$ and

$$e(u) = \frac{u}{\alpha + 2\delta \ln u} \quad \text{for all } u \geq 1$$

(see Embrechts et al. (1997)). In particular, $e(u)$ is eventually non-decreasing and condition (13) is satisfied for all $\gamma > 1$. Hence, the conditions of the proposition are met if (12) holds for some $\gamma > 1$. Again, this doesn't depend on γ, and is equivalent to $\overline{F}(u) = o(1/\ln u)$.

Finally, assume that $B(\cdot)$ is a Benktander type II distribution (see, for instance, Embrechts et al. (1997)) of the form:

$$\overline{B}(u) = \mathrm{e}^{\alpha/\delta} u^{-(1-\delta)} \exp\left\{-\frac{\alpha}{\delta} u^\delta\right\} \quad u \geq 1,\ 0 < \alpha < 1,\ 0 < \delta < 1.$$

In this case $B \in MDA(\Lambda)$, $B_0 \in \mathcal{S}$ and $e(u) = u^{1-\delta}/\alpha$, $u \geq 1$ (see, for instance, Embrechts et al. (1997)). Therefore $e(u)$ is non-decreasing and condition (13) is met for all $\gamma > 1/(1-\delta)$. As (12) can be rewritten as $\overline{F}(u) = o(u^{-\gamma\delta})$, the conditions of the proposition are satisfied if this holds for some $\gamma > 1/(1-\delta)$. This is the case if F is either light-tailed or has regularly varying tail of index $-\beta$ with $\beta > \delta/(1-\delta)$.

Proof of Proposition 3.1 In view of (6), it only remains to prove the corresponding lower bound. We have by (10) that

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \lim_{u\to\infty} \frac{\check{\Psi}_a((u+ca)/F(a))}{\overline{B}_0((u+ca)/F(a))} \lim_{u\to\infty} \frac{\overline{B}_0((u+ca)/F(a))}{\overline{B}_0(u)}. \tag{14}$$

Now, since $\overline{B} \in \mathcal{R}(-\alpha-1)$, it follows from Karamata's theorem (see, for instance, Embrechts et al. (1997)) that $\overline{B_0} \in \mathcal{R}(-\alpha)$. Hence, by the definition of regularly varying functions,

$$\lim_{u\to\infty} \frac{\overline{B}_0((u+ca)/F(a))}{\overline{B}_0(u)} = F(a)^\alpha. \tag{15}$$

Since $F(a) \leq 1$ for all a, it follows from the net profit condition that $\rho(a) := (F(a)\mu)/(c\nu) < 1$. Hence, by the classical ruin estimate,

$$\lim_{u\to\infty} \frac{\check{\Psi}_a((u+ca)/F(a))}{\overline{B}_0((u+ca)/F(a))} = \frac{\rho(a)}{1-\rho(a)}. \tag{16}$$

Substituting (15) and (16) in (14), we get

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \frac{\rho(a)F(a)^\alpha}{1-\rho(a)} \quad \text{for all } a > 0.$$

Letting a tend to infinity, we notice that $F(a)$ tends to 1 and $\rho(a)$ tends to ρ, which yields the desired lower bound, and the claim of the proposition.

□

Proof of Proposition 3.2 For part (a), note that if F has compact support, then there is an $a < \infty$ such that $F(a) = 1$. Consequently, we obtain from (14)

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \lim_{u\to\infty} \frac{\check{\Psi}_a(u+ca)}{\overline{B}_0(u+ca)} \lim_{u\to\infty} \frac{\overline{B}_0(u+ca)}{\overline{B}_0(u)}.$$

The first limit in the product above equals $\rho/(1-\rho)$ by the classical ruin estimate, while the second limit equals 1 by the long-tailed property of subexponential distributions (see, for instance, Embrechts et al. (1997)). Thus, we obtain a lower bound on $\psi(u)$ matching the asymptotic upper bound in (6) and equal to the limit in (11), as claimed.

Next, we turn to the proof of part (b). We can generalize (14) to let a depend on u, i.e., $a = a(u)$. Also note that for any $x > 0$ and $a \leq a(u)$, $\check{\Psi}_a(x) \leq \check{\Psi}_{a(u)}(x)$. This is obvious on recalling that $\check{\Psi}_a(x)$ denotes the ruin probability subject to initial capital x and premium rate $c/F(a)$, as the ruin probability is non-increasing in the premium rate. In other words, $\psi(u) \geq \check{\Psi}_{a(u)}((u+ca(u)))/F(a(u))) \geq \check{\Psi}_a((u+ca(u)))/F(a(u)))$. Therefore, we can rewrite (14) as

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \liminf_{u\to\infty} \frac{\check{\Psi}_a((u+ca(u)))/F(a(u)))}{\overline{B}_0((u+ca(u))/F(a(u)))} \liminf_{u\to\infty} \frac{\overline{B}_0((u+ca(u))/F(a(u)))}{\overline{B}_0(u)},$$

for any $a > 0$. Hence, using the classical ruin estimate applied to $\check{\Psi}_a$,

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \frac{\rho(a)}{1-\rho(a)} \liminf_{u\to\infty} \frac{\overline{B}_0((u+ca(u))/F(a(u)))}{\overline{B}_0(u)} \quad \text{for all } a > 0. \tag{17}$$

We now use a representation of $\overline{B}_0$ for $B \in MDA(\Lambda)$ given in Asmussen and Klüppelberg (1996):

$$\overline{B}_0(u) = \exp\Big(-\int_0^u \frac{1}{e(t)}\mathrm{d}t\Big), \quad u > 0.$$

It follows that

$$\frac{\overline{B}_0((u+ca(u))/F(a(u)))}{\overline{B}_0(u)} = \exp\Big(-\int_u^{\frac{u+ca(u)}{F(a(u))}} \frac{1}{e(t)}\mathrm{d}t\Big)$$

By the assumption of Proposition 3.2(b), $e(t) \sim g(t)$ for $g(\cdot)$ which is eventually non-decreasing. Hence, we have for all $\varepsilon > 0$ that

$$\liminf_{u\to\infty} \frac{\overline{B}_0((u+ca(u))/F(a(u)))}{\overline{B}_0(u)} \geq \liminf_{u\to\infty} \exp\Big(-\Big(\frac{u+ca(u)}{F(a(u))} - u\Big)\frac{1+\varepsilon}{g(u)}\Big). \tag{18}$$

Since $B \in MDA(\Lambda)$ and $B_0 \in \mathcal{S}$, the mean excess function $e(u)$ goes to ∞ as $u \to \infty$ (see Goldie and Resnick (1988)). Let $\gamma > 0$ be such that (12) and (13) hold, and take $a(u) = u^{1/\gamma}$. Then

$$\Big(\frac{u+ca(u)}{F(a(u))} - u\Big)\frac{1}{g(u)} = \frac{u\overline{F}(u^{1/\gamma})}{g(u)F(u^{1/\gamma})} + \frac{cu^{1/\gamma}}{g(u)F(u^{1/\gamma})} \to 0 \text{ as } u \to \infty,$$

since $g(u) \sim e(u)$. Therefore, we have by (18) and the fact that $\overline{B}_0(\cdot)$ is non-increasing that

$$\lim_{u\to\infty} \frac{\overline{B}_0((u + cu^{1/\gamma})/F(u^{1/\gamma}))}{\overline{B}_0(u)} = 1, \tag{19}$$

whenever γ satisfies (12) and (13). Substituting this in (17) yields

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \frac{\rho(a)}{1-\rho(a)}$$

for all $a > 0$. Now letting $a \to \infty$, and noting that $\rho(a) \to \rho$, we obtain that

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{B}_0(u)} \geq \frac{\rho}{1-\rho}.$$

Combined with the upper bound in (6), this yields (11).
□

3.2 The finite horizon case

Throughout this section we assume that $\{T_n\}_{n\geq 1}$, $T_0 = 0$, is a homogeneous Poisson process with intensity ν^{-1}. The following proposition holds:

Proposition 3.3

(a) If $\overline{B} \in \mathcal{R}(-\alpha - 1)$, $\alpha > 0$, then

$$\lim_{u\to\infty} \frac{\psi(u, uT)}{\psi(u)} = 1 - (1 + (1-\rho)T)^{-\alpha}. \tag{20}$$

(b) Suppose that the assumptions of Proposition 3.2(b) are satisfied for some $\gamma > 1$, and with the function $g(\cdot)$ being regularly varying at infinity. Then,

$$\lim_{u\to\infty} \frac{\psi(u, e(u)T)}{\psi(u)} = 1 - \mathrm{e}^{-(1-\rho)T}. \tag{21}$$

Remark 1 Note that $e(u) \sim g(u)$ with $g(\cdot)$ regularly varying at infinity is satisfied for all the examples considered earlier.

Remark 2 The proposition says that, starting with initial capital u and conditional on ruin occuring, the time to ruin scales like u or $e(u)$ under the assumptions of parts (a) and (b) respectively. More precisely, the time to ruin divided by u converges in distribution to a Pareto under the assumptions of part (a), while the time to ruin divided by $e(u)$ converges in distribution to an Exponential under the assumptions of part (b).

Proof We first show the upper bounds for (20) and (21). Denote, respectively, by $\Psi(u,T)$ and $\Psi(u)$ the finite horizon and the infinite horizon ruin probability relative to the classical risk process $\{C(t)\}_{t\geq 0}$ defined in (4). By Corollary 1.6 in Asmussen and Klüppelberg (1996) we have:

$$\lim_{u\to\infty} \frac{\Psi(u, uT)}{\Psi(u)} = 1 - (1 + (1-\rho)T)^{-\alpha} \tag{22}$$

if $\overline{B} \in \mathcal{R}(-\alpha - 1)$, and

$$\lim_{u\to\infty} \frac{\Psi(u, e(u)T)}{\Psi(u)} = 1 - \mathrm{e}^{-(1-\rho)T} \tag{23}$$

if $B \in MDA(\Lambda)$ and $B_0 \in \mathcal{S}$. By Propositions 3.1 and 3.2, $\psi(u) \sim \Psi(u)$ under the assumptions of this proposition. Since (5) implies that $\psi(u, \tilde{g}(u)T) \le \Psi(u, \tilde{g}(u)T)$ for all u and non-negative functions $\tilde{g}(\cdot)$, the upper bounds follow by (22) and (23).

It remains to prove the matching lower bounds. We first show part (a). For each $a > 0$, denote by $\check{\Psi}_a(u, T)$ and $\check{\Psi}_a(u)$ respectively the finite horizon and the infinite horizon ruin probability relative to the risk process $\{\check{C}_a(t)\}_{t\ge 0}$ defined by (8). As in the proof of Proposition 3.1, we get (9), which yields for each $a > 0$ that

$$\psi(u, uT) \ge \check{\Psi}_a((u+ca)/F(a), uT - a) \quad \text{for all } u > 0, \tag{24}$$

where $\check{\Psi}_a(u, t)$ is defined as zero if $t < 0$. Since $u - (a/T) \sim u + ca$ as $u \to \infty$, we have by (24) that for any $\varepsilon > 0$ there exists $\bar{u} = \bar{u}(\varepsilon)$ such that

$$\psi(u, uT) \ge \check{\Psi}_a((u+ca)/F(a), (u+ca)(1-\varepsilon)T) \quad \text{for all } u \ge \bar{u} \text{ and } a > 0. \tag{25}$$

Since (22) also holds with $\check{\Psi}_a$ in place of Ψ (the former is simply the ruin probability for the risk process modified to have premium rate $c/F(a)$), we get

$$\lim_{u\to\infty} \frac{\check{\Psi}_a((u+ca)/F(a), (u+ca)(1-\varepsilon)T)}{\check{\Psi}_a((u+ca)/F(a))} = 1 - (1 + (1-\rho(a))(1-\varepsilon)TF(a))^{-\alpha}, \tag{26}$$

where $\rho(a)$ is defined as $F(a)\mu/(c\nu) = \rho F(a)$. Arguing as in the proof of Proposition 3.1, we have

$$\lim_{u\to\infty} \frac{\check{\Psi}_a((u+ca)/F(a))}{\psi(u)} = \left(\frac{\rho(a)F(a)^\alpha}{1-\rho(a)}\right)\left(\frac{\rho}{1-\rho}\right)^{-1}. \tag{27}$$

The matching lower bound follows combining (25), (26), (27), letting ε tend to 0 and a tend to infinity.

Finally, we show part (b). We have from (9) that, for arbitrary positive functions $a(\cdot)$, arbitrary $a > 0$, and all $u > 0$ such that $a(u) \ge a$,

$$\psi(u, e(u)T) \ge \check{\Psi}_{a(u)}\Big(\frac{u + ca(u)}{F(a(u))}, e(u)T - a(u)\Big) \ge \check{\Psi}_a\Big(\frac{u + ca(u)}{F(a(u))}, e(u)T - a(u)\Big). \tag{28}$$

Recall that $e(\cdot)$ was assumed to be asymptotically equivalent to a regularly varying function $g(\cdot)$. Let β denote the index of variation of g at infinity. Let $\gamma > 1$ be such that (12) and (13) hold, and take $a(u) = u^{1/\gamma}$. Then, we have

$$\lim_{u\to\infty} \frac{e((u+ca(u))/F(a(u)))}{e(u)} = \lim_{u\to\infty} \Big(\frac{u + cu^{1/\gamma}}{uF(u^{1/\gamma})}\Big)^\beta = 1,$$

where we have used the fact that $\gamma > 1$ to obtain the second equality. Since $u^{1/\gamma} = o(e(u))$ by (13), it follows that

$$\lim_{u\to\infty} \frac{e(u)T - a(u)}{e((u+ca(u))/F(a(u)))} = T.$$

Therefore, by (23), we obtain for all $\varepsilon > 0$ that

$$\liminf_{u\to\infty} \frac{\check{\Psi}_a(\frac{u+ca(u)}{F(a(u))}, e(u)T - a(u))}{\check{\Psi}_a(\frac{u+ca(u)}{F(a(u))})} \ge 1 - \mathrm{e}^{-(1-\varepsilon)(1-\rho(a))T}.$$

Combining the above with (28) and letting ε decrease to zero, we get

$$\liminf_{u\to\infty} \frac{\psi(u, e(u)T)}{\check{\Psi}_a((u+ca(u))/F(a(u)))} \ge 1 - \mathrm{e}^{-(1-\rho(a))T}. \tag{29}$$

It remains only to compare the denominator of the left-hand side with $\Psi(u)$. Using the classical ruin estimate, we have

$$\liminf_{u\to\infty} \frac{\check{\Psi}_a((u+ca(u))/F(a(u)))}{\Psi(u)} = \frac{\rho(a)}{1-\rho(a)} \frac{1-\rho}{\rho} \liminf_{u\to\infty} \frac{\overline{B}_0((u+ca(u))/F(a(u))))}{\overline{B}_0(u)}.$$

If we take $a(u) = u^{1/\gamma}$ where $\gamma > 1$ satisfies the assumptions of Proposition 3.2(b), then (19) holds. Therefore,

$$\liminf_{u\to\infty} \frac{\check{\Psi}_a((u+cu^{1/\gamma})/F(u^{1/\gamma}))}{\Psi(u)} = \frac{\rho(a)}{1-\rho(a)} \frac{1-\rho}{\rho}.$$

Substituting this in (29), and using the asymptotic equivalence of $\psi(u)$ and $\Psi(u)$ established in Proposition 3.2, we get

$$\liminf_{u\to\infty} \frac{\psi(u, e(u)T)}{\psi(u)} \geq \frac{\rho(a)}{1-\rho(a)} \frac{1-\rho}{\rho} (1 - \mathrm{e}^{-(1-\rho(a))T}),$$

for all $a > 0$. Now letting a tend to infinity, and noting that $\rho(a)$ tends to ρ, we obtain the lower bound

$$\liminf_{u\to\infty} \frac{\psi(u, e(u)T)}{\psi(u)} \geq 1 - \mathrm{e}^{-(1-\rho)T}.$$

Combining this with the upper bound established earlier, the proof of the proposition is complete. □

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4 Appendix

In this section, we obtain logarithmic asymptotics for the infinite horizon ruin probability when the claim sizes have exponential tails and the claim arrival process is Poisson. As in Brémaud (2000), we consider the more general form of the risk process with delayed claims, namely

$$S(t) = \sum_{n \geq 1} H(t - T_n, Z_n) \mathbf{1}_{(0,t]}(T_n) - ct. \tag{30}$$

Here $T_0 = 0$ and $\{T_n\}_{n \geq 1}$ are the points of a Poisson process of rate λ, independent of the iid sequence of marks $\{Z_n\}_{n \geq 1}$, taking values in a measure space $(E, \mathcal{E})$, and $H : \mathbb{R}_+ \times E \to [0, \infty)$ is a measurable function such that $H(\cdot, z)$ is non-decreasing and cadlag (right continuous with left limits) for all $z \in E$. We define $H(\infty, z) = \lim_{t \to \infty} H(t, z)$, noting that the limit exists but may be infinite. The ruin probability $\psi(u)$ for $u > 0$ is defined as in (2).

We compare the risk process $S(\cdot)$ with the classical risk process $C(\cdot)$ defined by

$$C(t) = \sum_{n \geq 1} H(\infty, Z_n) \mathbf{1}_{(0,t]}(T_n) - ct, \tag{31}$$

with corresponding ruin probability $\Psi(u) = P(\sup_{t \geq 0} C(t) > u)$. We are interested in the case where the claim size distribution has exponentially decaying tails, i.e., $\mathrm{E}[\exp(\theta H(\infty, Z_1))]$ is finite for θ in a neighbourhood of 0. If Cramér's condition is satisfied, i.e.,

$$\exists\, w > 0 : \; \lambda\Big(\mathrm{E}\Big[e^{wH(\infty,Z_1)}\Big] - 1\Big) - cw = 0, \tag{32}$$

then we have the classical result

$$\lim_{u\to\infty} \frac{1}{u} \log \Psi(u) = -w. \tag{33}$$

It was shown in Brémaud (2000) that $\psi(u)$ satisfies the same logarithmic asymptotics, i.e.,

$$\lim_{u\to\infty} \frac{1}{u} \log \psi(u) = -w. \tag{34}$$

We now give an alternative proof of this result. Since $\psi(u) \le \Psi(u)$ for all $u > 0$, it suffices to prove the asympotic lower bound. To this end, observe that for all $a > 0$,

$$S(t) \ge \sum_{n\ge 1} H(a, Z_n)\mathbf{1}_{(0,t-a]}(T_n) - c(t-a) - ca = C_a(t-a) - ca,$$

where $C_a(\cdot)$ is defined analogous to $C(\cdot)$, but with $H(\infty, Z_1)$ replaced with $H(a, Z_1)$. Define Ψ_a to be the ruin probability associated with the risk process $C_a(\cdot)$. It follows from the above that

$$\psi(u) \ge \Psi_a(u + ca) \quad \text{for all } u > 0 \text{ and } a > 0. \tag{35}$$

We also have by the Cramér-Lundberg theorem that, for each $a > 0$,

$$\lim_{u\to\infty} \frac{1}{u} \log \Psi_a(u) = -w_a, \tag{36}$$

where w_a is the unique positive solution of

$$\lambda\Big(\mathrm{E}\Big[e^{w_a H(a,Z_1)}\Big] - 1\Big) - cw_a = 0$$

if one exists, and $w_a = \infty$ otherwise. Since $H(a, Z_1)$ increases to $H(\infty, Z_1)$ as $a \to \infty$, it readily follows that w_a decreases to w. Therefore, it is immediate from (35) and (36) that

$$\liminf_{u\to\infty} \frac{1}{u} \log \psi(u) \ge -\limsup_{u\to\infty} w_a = -w.$$

Combined with the upper bound established earlier, this completes the proof of (34).

Finally, we remark that the extension of the results to the case when $\{T_n\}_{n\ge 1}$ constitute a renewal process is straightforward. The results can also be extended to allow certain kinds of dependence between T_n and Z_n by following the methods of Albrecher and Teugels (2006). As the heavy-tailed case is the main focus of this paper, we do not pursue these extensions here.