Complex Networks
Problem Sheet 2

1. Let $X_t, t \geq 0$ be a Poisson process of rate λ, and let $c > 0$ be a given constant. Define $Y_t = X_{ct}$ for all $t \geq 0$. Use the answer to Question 5 from Homework 1 to show that $Y_t, t \geq 0$ is a Poisson process of rate $c\lambda$ by showing that the times between succesive increments of the Y_t process are iid $\text{Exp}(c\lambda)$ random variables.

2. Let $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ be independent Poisson processes of rate λ_1 and λ_2 respectively. Let $X_t = X_t^1 + X_t^2$ denote their superposition. Use the answer to Question 1 in Homework 1 to show that $X_t, t \geq 0$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$.

3. Let $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ be independent Poisson processes of rate λ_1 and λ_2 respectively. Let $X_t = X_t^1 + X_t^2$ denote their superposition. Use the answer to Question 5 in Homework 1 to show that $X_t, t \geq 0$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$ by showing that the times between successive increments of the X_t process are iid $\text{Exp}(\lambda)$ random variables.

4. We say that a random variable N has a Geometric distribution with parameter p, written $N \sim Geom(p)$ if
$$P(N = k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

   Let $N \sim Geom(p)$, and let $T_1, T_2, T_3, \ldots$ be iid $\text{Exp}(\lambda)$ random variables, independent of N. Let $T = \sum_{k=1}^{N} T_k$. Using moment generating functions or otherwise, show that T is exponentially distributed with parameter λp. (*Hint.* Recall that the moment generating function of T is defined as $M(\theta) = \mathbb{E}[\exp(\theta T)]$. First compute $\mathbb{E}[\exp(\theta T)|N = n]$ and then average over N to obtain the unconditional expectation.)

5. Let $X_t, t \geq 0$ be a Poisson process of rate λ_1, and let $Y_1, Y_2, Y_3, \ldots$ be iid Bernoulli(p) random variables. Recall that this means that $Y_i = 1$ with probability p and $Y_i = 0$ with probability $1 - p$.

   Let $X_t^1 = \sum_{i=1}^{X_t} Y_i$ be the process obtained by retaining each point of the Poisson process X_t independently with probability p and discarding it with probability $1 - p$. It is called the Bernoulli(p) thinning of the Poisson process X_t.

   Using the answer to question 4, show that $X_t^1, t \geq 0$ is a Poisson process of rate λp by showing that the times between successive events are $\text{Exp}(\lambda p)$.

6. Suppose that the rate matrix Q of a Markov chain is diagonalisable, and can be written as ADA^{-1}, where D is a diagonal matrix. Show that for $t > 0$, the transition probability matrix $P(t)$ can be written as
$$P(t) = Ae^{Dt}A^{-1}.$$