Complex Networks
Solutions 4

---

1. (a) The elements of the matrix P are given by $p_{ij} = 1/\text{deg}(i)$ if $(i,j) \in E$, and $p_{ij} = 0$ if $(i,j) \notin E$. Hence, P is stochastic.

   (b) The detailed (or local) balance equations say that $\pi_i p_{ij} = \pi_j p_{ji}$ for all i and j in V. Since the graph is undirected, this equation reads $0 = 0$ if there is no edge between i and j; this is trivially true. If there is an edge, this equation gives

$$\frac{\pi_i}{\text{deg}(i)} = \frac{\pi_j}{\text{deg}(j)}.$$

   It follows that $\pi_i = c\text{deg}(i)$, for some constant c, and for all i in the same connected component. It is possible to have different constants for different components. If we assume that the graph is connected, then a single constant c applies to all nodes in the graph, and the invariant distribution is unique; otherwise, there are infinitely many invariant distributions, of which this is one. To find the constant, we use the fact that the π_i should sum to 1. Hence,

$$c \sum_{i \in V} \text{deg}(i) = 1, \; i.e., \; c \cdot 2|E| = 1,$$

   where $|E|$ denotes the total number of edges in the graph. The sum of the degrees is $2|E|$ because each edge (i,j) gets counted twice in the sum, once in the degree of node i and once in that of j. Hence, we conclude that $\pi_i = \text{deg}(i)/(2|E|)$.

   (c) If the Markov chain is irreducible, it has a unique invariant distribution, i.e., a unique left eigenvector corresponding to the eigenvalue 1. In particular, this eigenvalue is not repeated, and there is a unique right eigenvector with eigenvalue 1. As the matrix P is stochastic, it is clear that the all-1 vector, denoted $\mathbf{1}$ is one such eigenvector. Hence, it is the only one.

   Aperiodicity guarantees that, starting from any initial distribution μ, the distribution at time t, which is given by μP^t, converges to the invariant distribution π as t tends to infinity. This says that all other eigenvalues are strictly smaller than 1 in absolute valuel. Consequently, for arbitrary initial conditions $\mathbf{x}(0)$, it follows that $\mathbf{x}(t) = P^t \mathbf{x}(0)$ converges to some multiple of the right eigenvector $\mathbf{1}$ corresponding to the largest eigenvalue in absolute value, namely 1. As we saw in lectures, this multiple is given by $\pi \mathbf{x}(0)$, i.e., $\mathbf{x}(t)$ tends to $(\pi \mathbf{x}(0))\mathbf{1}$ as t tends to infinity.

   (d) Substituting for π above, we see that $\mathbf{x}(t)$ tends to $1/(2|E|) \sum_{i \in V} \text{deg}(i) x_i(0) \mathbf{1}$ as t tends to infinity. Hence, the influence of a node on the final state is proportional to its degree.

2. (a) It is easy to check that $\mathbf{1}$, the vector that takes the value 1 on all $v \in V$ is an eigenvector of A_G with eigenvalue d. Likewise, the vector v_2 that takes the value 1 for all $v \in X$ and the value -1 for all $v \in Y$ is also easily seen to be an eigenvector of A_G, with eigenvalue $-d$. (If this is not obvious, observe that A_G has the following block form:

$$A_G = \begin{pmatrix} \mathbf{0} & A_{XY} \\ A_{XY} & \mathbf{0} \end{pmatrix},$$

where $A_{XY}(i,j) = 1$ if there is an edge between x_i and y_j, and 0 otherwise, and $\mathbf{0}$ denotes the all-zero block matrix of size $|X| \times |X|$. Note also that $|X| = |Y|$; this was not stated explicitly in the problem, but follows from the fact that the graph is bipartite and all nodes haves the same degree. To see this, count the number of edges between X and Y from each side.

Now, the matrix A_{XY} has each of its row sums equal to d because all nodes have degree d. Using this, it is easy to see that the vectors $\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{1} \\ -\mathbf{1} \end{pmatrix}$ are eigenvectors of A_G with eigenvalues $+d$ and $-d$ respectively.)

(b) One of the conclusions of the Perron-Frobenius theorem is that a positive matrix A satisfying its conditions has a positive principal eigenvalue which is *strictly* larger in absolute value than all other eigenvalues, and is the only one with a non-negative eigenvector. In this case, d has a non-negative eigenvector, the all-1 vector, so it is the principal eigenvalue. However, there is another eigenvalue $-d$ which is equally large in absolute value.

(c) The condition that is violated is that A_G^k be strictly positive for some k. All powers of A_G have the block structure

$$A_G^k = \begin{pmatrix} C_k & D_k \\ E_k & F_k \end{pmatrix}$$

where D_k and E_k are zero for even k, and C_k and F_k are zero for odd k. Hence, there is no k for which A_G^k is strictly positive.

3. (a) The rate matrix is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & \dots & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(b) Suppose $X_t \in \{1, 2, \dots, n-1\}$. Using the rates specified, we see that

$$\begin{aligned} &\mathbb{E}[M_{t+t+dt} - M_t | (X_u, u \le t)] \\ &= \lambda dt \Big(\Big(\frac{\mu}{\lambda}\Big)^{X_t+1} - \Big(\frac{\mu}{\lambda}\Big)^{X_t} \Big) + \mu dt \Big(\Big(\frac{\mu}{\lambda}\Big)^{X_t-1} - \Big(\frac{\mu}{\lambda}\Big)^{X_t} \Big) \\ &= \Big(\frac{\mu}{\lambda}\Big)^{X_t} (\mu - \lambda + \lambda - \mu) dt = 0 \end{aligned}$$

ignoring $o(dt)$ terms. Hence $\mathbb{E}[M_t]$ remains constant while $X_t \in \{1, \ldots, n-1\}$. Moreover, 0 and n are absorbing states, so X_t remains constant after hitting 0 or n. Hence, so does M_t and, consequently, its expectation. This completes the proof that M_t is a martingale. (You can also prove it by looking at the process at the jump times of X_t.)

(c) Let $T = \inf\{t \geq 0 : X_t = 0 \text{ or } n\}$. Then T is a stopping time, and we have by the Optional Stopping Theorem that

$$\mathbb{E}_k[M_T] = \left(\frac{\mu}{\lambda}\right)^n \mathbb{P}_k(X_T = n) + \left(\frac{\mu}{\lambda}\right)^0 \mathbb{P}_k(X_T = 0) = \mathbb{E}_k[M_0] = \left(\frac{\mu}{\lambda}\right)^k.$$

Noting that $\mathbb{P}_k(X_T = 0) = 1 - \mathbb{P}_k(X_T = n)$, we can rewrite the above as

$$\left(\left(\frac{\mu}{\lambda}\right)^n - 1\right)\mathbb{P}_k(X_T = n) = \left(\frac{\mu}{\lambda}\right)^k - 1,$$

and so

$$\mathbb{P}_k(X_T = n) = \frac{(\mu/\lambda)^k - 1}{(\mu/\lambda)^n - 1}.$$

4. (a) Each gene in generation $t+1$ is sampled independently and uniformly from the genes in the previous generation. Hence, each gene is an A allele with probability N_t/N, independent of all other genes. There are N genes again in generation $t+1$. Consequently, the number of A alleles in generation $t+1$ has a Bin$(N, N_t/N)$ distribution.

(b) It follows from the answer to the last part, and the fact that the mean of a Bin(n, p) distribution is np that the mean number of A alleles in generation $t+1$, conditional on the past, is N_t. More formally,

$$\mathbb{E}[N_{t+1} | N_t = k, N_0, N_1, \ldots, N_{t-1}] = \mathbb{E}[N_{t+1} | N_t = k] = N \cdot (k/N) = k.$$

The first equality above is due to the Markov property.

Thus, we have shown that $(N_t, t = 0, 1, 2, \ldots)$ is a discrete time martingale.

(c) Define $T = \inf\{t \geq 0 : N_t = 0 \text{ or } N\}$ to be the first time that all alleles are of the same type, either A or a. This is clearly a stopping time as we can decide whether $T \leq t$ only by knowing the process N_u for $u \leq t$. Moreover, N_t is a bounded martingale on $\{0, 1, 2, \ldots, T\}$, bounded between 0 and N. Hence, we can apply the optional stopping theorem. We obtain

$$\mathbb{E}[N_T] = N(0) = k.$$

But

$$\mathbb{E}[N_T] = \mathbb{P}(N_T = 0) \cdot 0 + \mathbb{P}(N_T = N) \cdot N.$$

Substituting this above, we conclude that

$$\mathbb{P}(N_T = N) = k/N.$$

This is the probability that only A alleles are left in the population.