Introduction to Queueing Networks

Problem Sheet 1

$**$ **Please hand in solutions to questions 4 and 5 on this sheet.** $**$

1. Ehrenfest urn model: There are n identical balls and two urns labelled A and B. At each time step, one of the n balls is chosen uniformly at random and moved to the other urn. Let X_t denote the number of balls in urn A after the t^{th} time step.

   (a) Briefly explain why $X_t, t \geq 0$ is a Markov chain. Draw an arrow diagram to describe the states and transition probabilities.

   (b) Specify its communicating classes and which states are transient and recurrent.

   (c) Compute all of its invariant distributions.

2. Wright-Fisher model of population genetics: A gene has two alleles (comes in one of two types) labelled A and a. For example, the gene could be for eye colour, allele A for brown eyes and allele a for blue eyes. Each individual has two copies of each gene (AA, Aa or aa) but here we are only interested in the number of copies of each allele in the population. The Wright-Fisher model is a highly simplified model assuming constant population size, non-overlapping generations, and random mating, and leads to the following description for the numbers of each allele.

   Let the total population size (of genes) be N, and let N_t denote the number of A alleles in generation t. Generation $t+1$ is obtained as follows. Each of the N genes in generation $t+1$ is sampled independently, and uniformly at random (with replacement), from the genes in generation t.

   (a) Briefly explain why $N_t, t \geq 0$, is a Markov chain. Write down the 1-step transition probability from state $N_t = n$ to state $N_{t+1} = n+3$ for $0 < n < n+3 \leq N$.

   (b) What are the communicating classes, transient states and recurrent states of your Markov chain?

   (c) What are its invariant distributions?

3. The weather in Bristol on any given day is either sunny, cloudy or rainy. Every sunny day is equally likely to be followed by either a sunny or a cloudy day, but never a rainy day. Every cloudy day is followed by a sunny, cloudy or rainy day with respective probabilites 0.4, 0.4 and 0.2. A rainy day is equally likely to be followed by a cloudy or rainy day, but never a sunny day.

   (a) Describe the weather in Bristol using a Markov chain, i.e., write down the states and the transition probabilities in the form of an arrow diagram.

(b) Identify the communicating classes and recurrent and transient states of this chain, and compute all its invariant distributions.

(c) Alice carries an umbrella with her if the day is either cloudy or rainy. How likely is it that Alice is carrying an umbrella today given that (i) she carried an umbrella yesterday, (ii) she carried an umbrella the last two days?

(d) Let Y_t be the indicator that Alice is carrying an umbrella on day t, i.e., $Y_t = 1$ if she is, and $Y_t = 0$ if she isn't carrying an umbrella on day t. Is $(Y_t, t \geq 0)$ a Markov chain?

4. Let X_1 and X_2 be independent Poisson random variables with means λ_1 and λ_2. Using generating functions, show that $X_1 + X_2$ is Poisson with mean $\lambda_1 + \lambda_2$.

5. (a) Let T have an Exponential distribution with parameter μ. Show that the distribution is "memoryless", i.e., show that for all $t, u > 0$,

$$P(T > t + u | T > u) = P(T > t).$$

(b) Let T_1 and T_2 be independent Exponential random variables with parameters λ_1 and λ_2 respectively, and let $T = \min\{T_1, T_2\}$.

i. Show that the distribution of T is Exp($\lambda_1 + \lambda_2$).

ii. Show that the probability that $T = T_1$ is $\lambda_1/(\lambda_1+\lambda_2)$, and that this is independent of the value of T.