Introduction to Queueing Networks

Problem Sheet 2

** Please hand in solutions to questions 3 and 5 on this sheet. **

1. Consider a continuous-time Markov chain $X_t$ on a finite state space $S$ with infinitesimal generator $Q$, and suppose that $X_0 = i$. There are two ways to describe the first jump out of this state.

- For each $j \neq i$ with $q_{ij} > 0$, let $T_j$ be an exponential random variable with parameter $q_{ij}$, and assume these random variables are mutually independent. Let $T = \min_j T_j$. Then the first jump occurs at the random time $T$, and $X_T = k$ if $T = T_k$. (This defines $X_T$ unambiguously because the probability that $T = T_j$ and $T = T_k$ for distinct $j$ and $k$ is zero.)

- Let $T$ be exponential with parameter $-q_{ii}$. Then the first jump occurs at the random time $T$ and $X_T = k$ with probability $q_{ik}/(-q_{ii})$, independent of the value of $T$.

Use the answer to Question 5 from Homework 1 to explain why these two descriptions are equivalent.

2. Let $X^1_t, t \geq 0$ and $X^2_t, t \geq 0$ be independent Poisson processes of rate $\lambda_1$ and $\lambda_2$ respectively. Let $X_t = X^1_t + X^2_t$ denote their superposition. Use the answer to Question 4 in Homework 1 to show that $X_t, t \geq 0$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$.

3. Let $X^1_t, t \geq 0$ and $X^2_t, t \geq 0$ be independent Poisson processes of rate $\lambda_1$ and $\lambda_2$ respectively. Let $X_t = X^1_t + X^2_t$ denote their superposition. Use the answer to Question 5 in Homework 1 to show that $X_t, t \geq 0$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$ by showing that the times between successive events of the $X_t$ process are iid Exp($\lambda$) (and any other properties required for it to be a Poisson process).

4. We say that a random variable $N$ has a Geometric distribution with parameter $p$, written $N \sim \text{Geom}(p)$ if

$$P(N = k) = p(1 - p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

Let $N \sim \text{Geom}(p)$, and let $T_1, T_2, T_3, \ldots$ be iid Exp($\lambda$) random variables, independent of $N$. Let $T = \sum_{k=1}^N T_k$. Using moment generating functions or otherwise, show that $T$ is exponentially distributed with parameter $\lambda p$. (Hint. Recall that the moment generating function of $T$ is defined as $M(\theta) = \mathbb{E}[\exp(\theta T)]$. First compute $\mathbb{E}[\exp(\theta T)|N = n]$ and then average over $N$ to obtain the unconditional expectation.)
5. Let $X_t, t \geq 0$ be a Poisson process of rate $\lambda_1$, and let $Y_1, Y_2, Y_3, \ldots$ be iid Bernoulli($p$) random variables. Recall that this means that $Y_i = 1$ with probability $p$ and $Y_i = 0$ with probability $1 - p$.

Let $X_t^1 = \sum_{i=1}^{X_t} Y_i$ be the process obtained by retaining each point of the Poisson process $X_t$ independently with probability $p$ and discarding it with probability $1 - p$. It is called the Bernoulli($p$) thinning of the Poisson process $X_t$.

Using the answer to question 4, show that $X_t^1, t \geq 0$ is a Poisson process of rate $\lambda p$ by showing that the times between successive events are Exp($\lambda p$), and any other properties required.

6. Suppose that the rate matrix $Q$ of a Markov chain is diagonalisable, and can be written as $ADA^{-1}$, where $D$ is a diagonal matrix. Show that for $t > 0$, the transition probability matrix $P(t)$ can be written as

$$P(t) = Ae^{Dt}A^{-1}.$$