Introduction to Queuing Networks

Problem Sheet 4

$**$ **Please hand in solutions to questions 2 and 4 on this sheet.** $**$

1. Customers arrive at a taxi rank singly, according to a Poisson process of rate λ, while taxis arrive according to a Poisson process of rate μ. These two Poisson processes are independent of each other. There is infinite waiting room for customers, but parking is available only for K taxis. If a taxi arrives when all parking spaces are full, it simply leaves. If a taxi arrives when a customer is waiting, or a customer arrives when a taxi is waiting, then one customer and one taxi depart at once (from the head of the queue). Hence, it is not possible for both customers and taxis to be waiting simultaneouly at the taxi rank.

   We assume that $\lambda < \mu$.

   (a) On average, how long does a customer have to wait before getting a taxi?

   (b) On average, how long does a taxi have to wait before getting a customer?

   (c) What happens to the answers to the above two questions as K, the number of available parking places for taxis, tends to infinity? Provide an intuitive explanation for your answer.

   Hints

   (i) Take the state to be the number of waiting customers minus the number of waiting taxis. Note that this state unambiguously identifies the number of each as both customers and taxis can't be waiting simultaneously.

   (ii) You may use the following formulas, which hold for all $\rho \in [0, 1)$ and all $K \geq 0$, without proof:

$$\sum_{j=0}^{\infty} j\rho^j = \frac{\rho}{(1-\rho)^2},$$
$$\sum_{j=0}^{K} j\rho^{-j} = \frac{K - (K+1)\rho^K + \rho^{K+1}}{\rho^K(1-\rho)^2}.$$

2. Consider a queue into which customers arrive according to a Poisson process of rate λ. The queue has a single server, and the service rate is $\mu(n+1)/n$ when there are n customers in the system. Let N_t denote the number of customers in the system at time t, and note from the description that N_t is a Markov process.

   (a) Depict all the possible transitions of the Markov process N_t and their rates, either in the form of an arrow diagram or in the form of a transition rate matrix.

(b) Determine the invariant distribution of the Markov process N_t, stating any condition on λ and μ required for one to exist.
Hint: The Taylor expansion of $-\log(1-x)$ is $x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$, for $0 \leq x < 1$.

(c) Using generating functions or otherwise, compute the mean queue length in the system, and the mean sojourn time of a customer.

3. Customers arrive at a restaurant singly, according to a Poisson process of rate λ. Customers prefer crowded restaurants as they perceive it as a sign of its quality. If there are already n people in the restaurant, then the probability that the customer will choose to go in is $\frac{n+1}{n+2}$. The restaurant has infinite capacity, and each customer spends a random amount of time in the restaurant, which has an $Exp(\mu)$ distribution and is independent of everything else (the customer arrival process, the number of people in the restaurant and the times spent by other customers).

   (a) Find the invariant distribution of the number of customers in the restaurant, stating any assumptions that λ and μ may have to satisfy in order for there to be an invariant distribution.

   (b) Find the distribution of the number of customers in the restaurant as seen by a typical customer who chooses to enter it.

4. Consider a single server queue with Poisson arrivals, where the service times are iid with a shifted Pareto distribution, whose density function is

$$f_S(x) = \frac{3}{(1+x)^4} 1(x \geq 0).$$

   (a) What is the largest arrival rate for which the queue is stable?

   (b) Suppose the arrival rate is $\lambda = 0.5$ and that the server employs a first-come-first-served (FCFS) service discipline. Use the Pollaczek-Khinchin formula to compute the mean number of customers in the system, and the mean sojourn time of a customer.

   (c) Suppose next that server adopts the processor-sharing (PS) service discipline. Compute the mean queue length and mean sojourn time in this case.

   (d) Comparing your answers to parts (b) and (c), which of these service disciplines would you recommend? How would your answer change if the service times are deterministic with the same mean? Give an intuitive justification for your answers.

5. Customers enter a cafetaria at a mean rate of one per minute during the lunch hour. Customers arrive and depart singly, and the arrivals form a Poisson process. Each customer spends a time uniformly distributed between 20 and 30 minutes eating lunch. The cafetaria owner wants to ensure that no more than 1% of customers are turned away due to a lack of space. How many dining spaces must the cafetaria provide?

   If you wish, you may use the fact that, if Z is a standard normal random variable, then $P(Z > 2.23) < 0.01 < P(Z > 2.22)$. You may also use the fact (which I encourage you to verify) that, if X is a Poisson(λ) random variable, then the mean and variance of X are both equal to λ.