Introduction to Queueing Networks

Solutions to Problem Sheet 2

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1. Question 5 from Homework 1 can be generalised to more than two exponential random variables (by direct calculation or by induction). If $T_1, T_2, \ldots, T_n$ are independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, and $T = \min_{i=1}^n T_i$, then $T \sim \text{Exp}(\lambda)$, where $\lambda = \sum_{i=1}^n \lambda_i$. Moreover, $T = T_i$ with probability λ_i/λ, independent of the value of T.

   Now suppose that T and X_T are defined as in the first part of the question. Then, by the above, $T \sim \text{Exp}(\sum_{j \in S} q_{ij})$. But $\sum_{j \in S} q_{ij} = -q_{ii}$ because the rows of the Q matrix sum to zero. Moreover, $P(T = T_k) = q_{ik}/(-q_{ii})$. This is exactly the same as the distribution of T and the probability that $X_T = k$ defined in the second part of the question. This shows that the two descriptions are equivalent.

2. As $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ are independent Poisson processes, we have for all $s, t > 0$ that $X_{t+s}^1 - X_s^1$ and $X_{t+s}^2 - X_s^2$ are independent Poisson random variables, with means $\lambda_1 t$ and $\lambda_2 t$ respectively (by one of the definitions of a Poisson process). Hence, by Question 4 of Homework 1, $X_{t+s} - X_s$ is a Poisson random variable with mean $(\lambda_1 + \lambda_2)t$.

   Moreover, the increments of the X^1 process are independent of the past of the X^1 process because it is a Poisson process, and independent of the past of the X^2 process, because these processes are mutually independent. Hence, the increments of the X^1 process are independent of the past of the X process. Likewise for the increments of the X^2 process. Putting these together, the increments of the X process are independent of the past of the X process. This is the other property we need to complete the proof that $X_t, t \geq 0$ is a Poisson process.

3. Denote the successive events in the Poisson processes $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ by $T_n^1, n \in \mathbb{N}$ and $T_n^2, n \in \mathbb{N}$ respectively, and the events in the superposition $X_t, t \geq 0$ by $T_n, n \in \mathbb{N}$. What can we say about T_1, the time until the first event in the superposition of the two Poisson processes? Clearly, it is the minimum of T_1^1 and T_1^2. Since X^1 and X^2 are Poisson processes, T_1^1 and T_1^2 are exponentially distributed with parameters λ_1 and λ_2 respectively. Moreover, T_1^1 and T_1^2 are independent random variables since the corresponding Poisson processes are independent. Hence, by the answer to HW1, Question 5, T_1 is exponentially distributed with parameter $\lambda = \lambda_1 + \lambda_2$.

   Next, irrespective of whether $T = T_1$ or T_2, the times until the next event of the two Poisson processes are exponentially distributed with parameters λ_1 and λ_2, independent of each other and of the past of the X^1 and X^2 processes (using the memoryless property of the exponential distribution). Hence, by the same reasoning, $T_2 - T_1$ is also an Exp(λ) random variable. The same reasoning applies to subsequent event times in the Poisson process $X_t, t \geq 0$.

   In order to complete the proof that X_t is a Poisson process of rate λ, we need to show that $X_{t+u} - X_t$ is independent of $X_s, s \leq t$ for arbitrary $t, u > 0$. The corresponding property is true of each of the processes X_t^1 and X_t^2 since these are Poisson processes. Moreover $X_{t+u}^2 - X_t^2$ is independent of $X_s^1, s \leq t$, and the same with superscripts 1 and 2 interchanged, since the X^1 and X^2 processes are independent of each other. Summing X^1 and X^2, it follows that the required independence properties hold for the X_t process.

4. First, the moment generating function of each T_i is given by

$$\begin{aligned} M_i(\theta) &:= E\Big[e^{\theta T_i}\Big] = \int_0^\infty e^{\theta t}\lambda e^{-\lambda t}dt \\ &= \begin{cases} \frac{\lambda}{\lambda-\theta}, & \text{if } \theta < \lambda, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, we obtain the conditional expectation

$$E\Big[e^{\theta T} \;\Big|\; N=n\Big] = E\Big[e^{\theta(T_1+T_2+\ldots+T_n} \;\Big|\; N=n\Big] = \prod_{i=1}^n E\Big[e^{\theta T_i}\Big].$$

To obtain the last equality above, we have used the fact that the T_i are mutually independent, and independent of the random variable N; hence, we can drop the conditioning on N, and replace the expectation of the product of $e^{\theta T_i}$ by the product of their expectations. Now, substituting for $M_i(\theta)$ above, we get,

$$E\Big[e^{\theta T} \;\Big|\; N=n\Big] = M_i(\theta)^n,$$

i.e.,

$$E\Big[e^{\theta T} \;\Big|\; N\Big] = M_i(\theta)^N = \begin{cases} \left(\frac{\lambda}{\lambda-\theta}\right)^N, & \text{if } \theta < \lambda, \\ +\infty & \text{otherwise.} \end{cases} \qquad (1)$$

Next, observe that the generating function of the random variable N is given by

$$G_N(z) := E\Big[z^N\Big] = \sum_{k=1}^\infty p(1-p)^{k-1}z^k = \begin{cases} \frac{pz}{1-(1-p)z}, & \text{if } z < 1/p, \\ +\infty, & \text{otherwise.} \end{cases}$$

Hence, on taking expectations with respect to N in (1), we obtain the moment generating function of T as

$$\begin{aligned} M_T(\theta) &:= E\Big[e^{\theta T}\Big] = E\Big[E\Big[e^{\theta T}|N\Big]\Big] \\ &= G_N(M_i(\theta)) = \begin{cases} \left(\frac{\lambda p}{\lambda-\theta}\right) \Big/ \left(1 - \frac{\lambda(1-p)}{\lambda-\theta}\right), & \text{if } \theta < \lambda \text{ and } \frac{\lambda}{\lambda-\theta} < \frac{1}{p}\,, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Simplifying the above expression, we finally have

$$M_T(\theta) = \begin{cases} \frac{\lambda p}{\lambda p-\theta}, & \text{if } \theta < \lambda p, \\ +\infty, & \text{otherwise.} \end{cases}$$

We recognise this as the moment generating function of an Exp(λp) random variable. Hence, using the fact that there is a one-to-one correspondence between probability distributions and moment generating functions (which we shall take for granted without proof), we conclude that T is an exponential random variable with parameter λp.

5. Let $T_1, T_2, \ldots$ be the times of successive events in the Poisson process $X_t, t \geq 0$, and let $T_1^1, T_2^1, \ldots$ denote the same for the process $X_t^1, t \geq 0$. Then,

$$\begin{aligned} P(T_1^1 \geq t) &= \sum_{n=1}^\infty P(T_n \geq t, Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) \\ &= \sum_{n=1}^\infty P(Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) P(T_n \geq t | Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) \\ &= \sum_{n=1}^\infty P(Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) P(T_n \geq t), \end{aligned}$$

since the Y_i are independent of the Poisson process $X_t, t \geq 0$.

Now, the first i for which $Y_i = 1$ is a geometric random variable, and $T_{i+1} - T_i$ for successive i are independent Exp(λ) random variables. Hence, T_1^1 is the sum of a Geom(p) number of iid copies of an Exp(λ) random variable. Hence, by the answer to Question 4, T_1^1 is an Exp(λ_p) random variable. The same argument applies to $T_2^1 - T_1^1, T_3^1 - T_2^1$ and so on, which are also clearly mutually independent (due to the mutually independence of times between events in the X_t process, and the fact that the Bernoulli sequence Y_i is iid). To show that $X_t^1, t \geq 0$ is a Poisson process of rate λp, it remains only to show that $X_{t+u}^1 - X_t^1$ is independent of $X_s^1, s \leq t$. But this is obvious from the discussion above.

6. Recall that $P(t) = e^{Qt}$. Using the diagonalisation of Q, we have

$$\begin{aligned} P(t) &= e^{Qt} = I + Qt + \frac{Q^2t^2}{2!} + \frac{Q^3t^3}{3!} + \dots \\ &= I + A(Dt)A^{-1} + A\frac{(Dt)^2}{2!}A^{-1} + A\frac{(Dt)^3}{3!}A^{-1} + \dots \\ &= A\Big(I + Dt + \frac{(Dt)^2}{2!} + \frac{(Dt)^3}{3!} + \dots\Big)A^{-1} = Ae^{Dt}A^{-1}. \end{aligned}$$

The point of this is that powers of a diagonal matrix simply correspond to taking the powers of the diagonal entries, element by element. Consequently, e^{Dt} is also a diagonal matrix, with jj^{th} element equal to $e^{d_{jj}t}$.