Introduction to Queuing Networks

Solutions to Problem Sheet 5

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1. (a) Consider a unit rate Poisson process. The number of points on the interval $[0, \lambda]$ has the same distribution as the random variable N. A Bernoulli p thinning of this Poisson process is a Poisson process of rate p. Moreover, the points of the thinned process on the interval $[0, \lambda]$ have the same distribution as the random variable Y. Hence, Y is a Poisson random variable with mean λp.

   (b) What can we say about the number of pairs of skis outstanding on day 0, which we shall denote X_0. Letting Y_t denote the number of pairs borrowed on day $t \leq 0$ but not yet returned by day 0, it is clear that we can express X_0 as

$$X_0 = \sum_{t=-\infty}^{0} Y_t = \sum_{t=-7}^{0} Y_t,$$

   since any skis rented before day -7 must have been returned by day 0.

   Let N_t denote the total number of pairs of skis borrowed out on day t, and let p_t denote the probability that a pair borrowed on day t has not yet been returned by day 0. Then, it is easy to see that Y_t is Binomial with parameters N_t and p_t. Moreover, we are given that N_t is Poisson with mean 100. Hence, by the answer to part (a), Y_t has a Poisson distribution with parameter $100p_t$.

   Also note that the numbers of skis borrowed on different days are independent random variables, and that the returning decisions for different people are mutually independent. Hence, Y_t for different t are mutually independent. Now, using the fact that the sum of independent Poisson random variables is Poisson with the sum of the parameters, it follows that X_0 is a Poisson random variable with parameter

$$\sum_{t=-7}^{0} 100p_t = 100\Big(\sum_{t=-7}^{-3} p_t + \sum_{t=-2}^{0} p_t\Big).$$

   Since skis are equally likely to be returned any day from 3 to 7 days after they are borrowed, the probability p_t that skis borrowed on day t have not been returned by day 0 is given by $p_t = (t+7)/5$ for $-7 \leq t \leq -3$ and by $p_t = 1$ for $t = 0, -1$, or -2. Hence, we get

$$\sum_{t=-7}^{-3} p_t + \sum_{t=-2}^{0} p_t = 2 + 3 = 5,$$

   and so the number of pairs of skis outstanding on day 0, X_0, is Poisson with mean 500.

   Note that this is the same as the mean number of rentals 100 per day, times the mean number of days for which the skis are kept, 5. Compare this with the results for the invariant queue length distribution in an $M/G/\infty$ queue. The point of this problem is that the model here is a discrete-time analogue of the $M/G/\infty$ queue.

2. (a) The possible transitions and rates are:

$$q(\mathbf{n}, \mathbf{n} + e_i) = \lambda_i, \quad q(\mathbf{n}, \mathbf{n} - e_i) = \mu_i n_i, \quad i = 1, 2,$$

   where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(b) The detailed balance equations are

$$\lambda_i \pi(\mathbf{n}) = \mu_i(n_i+1)\pi(\mathbf{n}+e_i), \quad i=1,2.$$

Consider the distribution

$$\pi(n_1,n_2) = \rho_1^{n_1}\frac{e^{-\rho_1}}{n_1!}\rho_2^{n_2}\frac{e^{-\rho_2}}{n_2!} \quad n_1=0,1,2,\dots, \quad n_2=0,1,2,\dots$$

where $\rho_i = \lambda_i/\mu_i,\ i=1,2$. It is easy to see that this is a probability distribution. Moreover,

$$\begin{aligned}\mu_1(n_1+1)\pi(n_1+1,n_2) &= \mu_1(n_1+1)\rho_1^{n_1+1}\frac{e^{-\rho_1}}{(n_1+1)!}\rho_2^{n_2}\frac{e^{-\rho_2}}{n_2!}\\ &= \mu_1\rho_1\rho_1^{n_1}\frac{e^{-\rho_1}}{n_1!}\rho_2^{n_2}\frac{e^{-\rho_2}}{n_2!} = \lambda_1\pi(n_1,n_2).\end{aligned}$$

Similarly, $\mu_2(n_2+1)\pi(n_1,n_2+1) = \lambda_2\pi(n_1,n_2)$. Hence, π satisfies the detailed balance equations. Therefore, the joint queue length process is reversible with invariant distribution π.

3. (a) The external arrival rates are $\gamma_1 = 1$, $\gamma_j = 0$, $j=2,\dots,J$. The routing probabilities for $1 \le j \le J-1$ are $r_{j,j+1} = (J-j)/J$ and $r_{j,j-1} = j/J$, while $r_{J,J-1} = 1$.

(b) The traffic equations are as follows:

$$\begin{aligned}\lambda_1 &= 1 + \frac{2}{J}\lambda_2,\\ \lambda_j &= \frac{J-j+1}{J}\lambda_{j-1} + \frac{j+1}{J}\lambda_{j+1}, \quad 2 \le j \le J-1,\\ \lambda_J &= \frac{1}{J}\lambda_{J-1}.\end{aligned}$$

To verify that $\lambda_j = \binom{J}{j}$ satisfies these equations, observe that

$$\begin{aligned}1 + \frac{2}{J}\binom{J}{2} &= 1 + J - 1 = J = \binom{J}{1},\\ \frac{J-j+1}{J}\binom{J}{j-1} + \frac{j+1}{J}\binom{J}{j+1} &= \frac{(J-1)!}{(J-j)!(j-1)!} + \frac{(J-1)!}{(J-j-1)!j!}\\ &= \frac{(J-1)!}{(J-j)!j!}(j+J-j) = \binom{J}{j},\ 2 \le j \le J-1,\\ \frac{1}{J}\binom{J}{J-1} &= 1 = \binom{J}{J}.\end{aligned}$$

(c) Observe that

$$\frac{\lambda_{j+1}}{\lambda_j} = \frac{J!}{(j+1)!(J-j-1)!}\frac{j!(J-j)!}{J!} = \frac{J-j}{j+1},$$

which is bigger than 1 if $j < (J-1)/2$ and smaller than 1 if $j > (J-1)/2$. Thus, λ_j is increasing for $j \le (J-1)/2+1$ and decreasing for $j > (J-1)/2$, which means that it attains its maximum value at $j = (J-1)/2$ (and $j = (J+1)/2$).

Hence, for the network to be stable, we need $\mu > \binom{J}{(J-1)/2}$. For $J=5$, this corresponds to $\mu > \binom{5}{2} = 10$.

4. (a) The traffic equations are as follows:

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{1}{4}\lambda_2, \\ \lambda_j &= \frac{1}{2} + \frac{3}{4}\lambda_{j-1} + \frac{1}{4}\lambda_{j+1}, \quad 2 \le j \le J_1, \\ \lambda_J &= \frac{J+3}{4} + \frac{3}{4}\lambda_{J-1}.\end{aligned}$$

Given that $\lambda_1 = 1$, the first equation above implies that $\lambda_2 = 4(1/ - \frac{1}{2}) = 2$. Then, taking $j = 2$ in the second equation above yields $\lambda_3 = 4(2 - \frac{1}{2} - \frac{3}{4} \cdot 1) = 3$.

(b) This leads us to guess that $\lambda_j = j$. It clearly satisfies the general equation for λ_j, since

$$\frac{1}{2} + \frac{3}{4}(j-1) + \frac{1}{4}(j+1) = j.$$

We also have $\frac{J+3}{4} + \frac{3}{4}(J-1) = J$, so the equation for J is satisfied as well.
Since $\lambda_j = j$ for $j = 1, 2, \ldots, J$, and we need $\mu > \lambda_j$ for all j for the network to be stable, we need $\mu > J$.

5. (a) The external arrival rates to the two queues are $\eta_1 = \lambda$ and $\eta_2 = 0$, while the routing parameters are $r_{12} = r$, $r_{10} = 1 - r$ and $r_{20} = 1$. By solving the traffic equations or directly, we can see that the total arrival rates into the two queues are $\lambda_1 = \lambda$ and $\lambda_2 = \lambda r$. Since both queues are infinite-server queues with service rate μ, the transition rates are as follows:

$$q(\mathbf{n}, \mathbf{n}+e_1) = \lambda,\; q(\mathbf{n}, \mathbf{n}-e_1) = \mu n_1(1-r),\; q(\mathbf{n}, \mathbf{n}-e_1+e_2) = \mu n_1 r,\; q(\mathbf{n}, \mathbf{n}-e_2) = \mu n_2.$$

Here $\mathbf{n} = (n_1 n_2)$ denotes the number of customers in the two queues, and e_1 and e_2 are the unit vectors (10) and (01) respectively.

(b) In reverse time, the external arrival rate into Q_2 is $\eta_2' = \lambda r$ since this is the total arrival rate into Q_2, and hence also the departure rate from Q_2 to outside, in forward time. Likewise, the external arrival rate into Q_1 in reverse time is $\eta_1' = \lambda(1-r)$, and the routing parameters are $r_{21}' = 1$ and $r_{10}' = 1$. Thus, the transition rates for the time-reversed Markov chain are

$$q'(\mathbf{n}, \mathbf{n}+e_1) = \lambda(1-r),\; q'(\mathbf{n}, \mathbf{n}+e_2) = \lambda r,\; q'(\mathbf{n}, \mathbf{n}-e_1) = \mu n_1,\; q'(\mathbf{n}, \mathbf{n}+e_1-e_2) = \mu n_2.$$

(c) We guess that the invariant distribution is product-form, with each queue having the invariant distribution corresponding to an $M/M/\infty$ queue, namely a Poisson distribution. In other words, we guess that the invariant distribution is

$$\pi(n_1, n_2) = \frac{\rho_1^{n_1}\rho_2^{n_2}}{n_1! n_2!} e^{-(\rho_1+\rho_2)},$$

where $\rho_1 = \lambda_1/\mu = \lambda/\mu$ and $\rho_2 = \lambda_2/\mu = \lambda r/\mu$. Thus, defining $\rho = \lambda/\mu$, we can rewrite the above as

$$\pi(n_1, n_2) = \frac{\rho^{n_1+n_2} r^{n_2}}{n_1! n_2!} e^{-\rho(1+r)},$$

To check the conditions of Kelly's lemma, we first check that the total transition rate out of any state $\mathbf{n}$ is the same in forward and reversed time. In forward time,

$$q_{\mathbf{n}} = q(\mathbf{n}, \mathbf{n}+e_1) + q(\mathbf{n}, \mathbf{n}-e_1) + q(\mathbf{n}, \mathbf{n}-e_1+e_2) + q(\mathbf{n}, \mathbf{n}-e_2) = \lambda + \mu(n_1+n_2),$$

whereas in reversed time,

$$q'_{\mathbf{n}} = q'(\mathbf{n}, \mathbf{n} + e_1) + q'(\mathbf{n}, \mathbf{n} + e_2) + q'(\mathbf{n}, \mathbf{n} + e_1 - e_2) + q'(\mathbf{n}, \mathbf{n} - e_1) = \lambda + \mu(n_1 + n_2).$$

These are equal and so the first condition of Kelly's lemma is satisfied.
Next, we need to check that for all states $\mathbf{n}$ and $\mathbf{m}$, $\pi(\mathbf{n})q(\mathbf{n}, \mathbf{m}) = \pi(\mathbf{m})q'(\mathbf{m}, \mathbf{n})$. Consider, for example, $\mathbf{m} = \mathbf{n} - e_1 + e_2$. In this case,

$$\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - e_1 + e_2) = \frac{\rho^{n_1+n_2} r^{n_2}}{n_1! n_2!} e^{-\rho(1+r)} \mu n_1 r = \frac{\rho^{n_1-1+n_2} r^{n_2+1}}{(n_1 - 1)! n_2!} e^{-\rho(1+r)} \lambda 1(n_1 > 0),$$

while

$$\begin{aligned} \pi(\mathbf{n} - e_1 + e_2)q'(\mathbf{n} - e_1 + e_2, \mathbf{n}) &= \frac{\rho^{n_1-1+n_2+1} r^{n_2+1}}{(n_1 - 1)!(n_2 + 1)!} e^{-\rho(1+r)} 1(n_1 > 0)\mu(n_2 + 1) \\ &= \frac{\rho^{n_1-1+n_2} r^{n_2+1}}{(n_1 - 1)! n_2!} e^{\rho(1+r)} \lambda 1(n_1 > 0). \end{aligned}$$

Thus, $\pi(\mathbf{n})q(\mathbf{n}, \mathbf{n} - e_1 + e_2) = \pi(\mathbf{n} - e_1 + e_2)q'(\mathbf{n} - e_1 + e_2, \mathbf{n})$ for all $\mathbf{n}$. The other equalities of Kelly's lemma can be checked similarly.