Elements of Linear Programming
Solution to Problem Sheet

1. To convert inequality constraints into equality constraints, non-negative slack variables w_1, w_2 are introduced. w_1 is added to the left-hand side of the first constraint, while w_2 is subtracted from the left-hand side of the second constraint. Non-positive variable x_3 is replaced by $-u_3$, where u_3 is a non-negative slack variable. Sign-free variables x_2, x_4 are replaced by $u_2 - v_2$, $u_4 - v_4$ (respectively), where u_4, v_4 are non-negative slack variable. As a result, we get the following LP:

$$\begin{aligned}
&\text{Maximize } x_1 + 3u_2 - 3v_2 - 2u_3 + u_4 - v_4 \\
&\text{Subject to} \\
&\quad x_1 - 2u_2 + 2v_2 - u_3 + u_4 - v_4 + w_1 = 4, \\
&\quad -x_1 + 3u_2 - 3v_2 + u_3 + 2u_4 - 2v_4 - w_2 = 5, \\
&\quad x_1 + u_2 - v_2 - u_3 + u_4 - v_4 = 10, \\
&\quad x_1 \geq 0,\ u_2 \geq 0, u_3 \geq 0, u_4 \geq 0, v_2 \geq 0, v_4 \geq 0, w_1 \geq 0, w_2 \geq 0.
\end{aligned}$$

2. (a) The given optimisation problem can be formulated as the following LP problem:

$$\begin{aligned}
&\text{Minimize } \ z \quad \text{(with respect to } x_1, \dots, x_n, z \in \mathbb{R}) \\
&\text{Subject to} \\
&\quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad \text{for } i = 1, \dots, m, \\
&\quad \sum_{j=1}^{n} c_{kj} x_j - z \leq -d_k, \quad \text{for } k = 1, \dots, p, \\
&\quad \sum_{j=1}^{n} c_{kj} x_j + z \geq -d_k, \quad \text{for } k = 1, \dots, p, \\
&\quad x_j \geq 0, \quad \text{for } j = 1, \dots, n.
\end{aligned}$$

To get the above LP problem set

$$z = \max_{1 \leq k \leq p} \left| \sum_{j=1}^{n} c_{kj} x_j + d_k \right|$$

and notice that

$$-z \leq \sum_{j=1}^{n} c_{kj} x_j + d_k \leq z$$

for $k = 1, \ldots, p$.

(b) The given optimisation problem can be formulated as the following LP problem:

$$\text{Minimize} \quad \sum_{k=1}^{p} z_k \quad (\text{with respect to } x_1, \ldots, x_n, z_1, \ldots, z_p \in \mathbb{R})$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad \text{for } i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} c_{kj} x_j - z_k \leq -d_k, \quad \text{for } k = 1, \ldots, p,$$

$$\sum_{j=1}^{n} c_{kj} x_j + z_k \geq -d_k, \quad \text{for } k = 1, \ldots, p,$$

$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

To get the above LP problem set

$$z_k = \left| \sum_{j=1}^{n} c_{kj} x_j + d_k \right|$$

for $k = 1, \ldots, p$, and notice that

$$-z_k \leq \sum_{j=1}^{n} c_{kj} x_j + d_k \leq z_k$$

for $k = 1, \ldots, p$.

3. (a) The feasible domain D and the objective level lines (sets) $OLL(c)$ are represented on Figure 1. From the graphical representation, we conclude that the objective level line going through V_0 is the objective level line which intersects the feasible domain (D) and which has the minimum value of c. Hence, the coordinates of V_0 are an optimal solution. These coordinates are $x = 1/2$, $y = 0$.

(b) The feasible domain D and the objective level lines (sets) $OLL(c)$ are represented on Figure 2. From the graphical representation, we conclude that the objective level line going through V_3 is the objective level line which intersects the feasible domain (D) and which has the minimum value of c. Hence, the coordinates of V_3 are an optimal solution. These coordinates are $x = 1$, $y = -1/3$.

(c) As $H_1 \cap H_2$ does not have a non-empty intersection with $\{(x, y) : x \leq 0\}$, the feasible domain is empty (see Figure 3). Hence, the problem is infeasible.

(d) The feasible domain D and the objective level lines (sets) $OLL(c)$ are represented on Figure 4. From the graphical representation, we conclude that the objective level line $OLL(c)$ intersects feasible domain D for any $c \in \mathbb{R}$. Therefore, the problem does not have a finite optimal solution, i.e., the optimal objective value is $-\infty$.

(e) The feasible domain D and the objective level lines (sets) $OLL(c)$ are represented on Figure 5. From the graphical representation, we conclude that the objective level line going through V_2 and V_3 is the objective level line which intersects the feasible domain (D) and which has the minimum value of c. Therefore, the coordinates of any point on the segment connecting V_2 and V_3 are an optimal solution. Hence,

$$x = 12\lambda, \quad y = 14(1-\lambda)$$

is an optimal solution for each $\lambda \in [0, 1]$.

4. Vector representation of the given problem:

$$\text{Minimize } c^T \boldsymbol{x} \tag{1}$$
$$\text{Subject to } A\boldsymbol{x} = b, \boldsymbol{x} \geq 0 \tag{2}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$$
$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
$$b = \begin{pmatrix} 6 & 3 \end{pmatrix}^T$$
$$c = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T$$

Possible combinations of basic indices:

$$B_1 = \{1,2\}, \quad B_2 = \{1,3\}, \quad B_3 = \{1,4\}, \quad B_4 = \{2,3\}, \quad B_5 = \{2,4\}, \quad B_6 = \{3,4\}$$

Basic Indices: $B_1 = \{1, 2\}$.

Matrix of Basic Columns and Its Determinant:

$$A_{B_1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \det A_{B_1} = 1 \neq 0 \tag{3}$$

Vector of Basic Variables: $\boldsymbol{x}_{B_1} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$.

Set $x_3 = x_4 = 0$ and compute

$$\hat{\boldsymbol{x}}_{B_1} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = A_{B_1}^{-1} b = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \geq \mathbf{0} \tag{4}$$

Basic Feasible Solution Relative to B_1:

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

Cost at $\hat{\boldsymbol{x}}_1$: $c^T\hat{\boldsymbol{x}}_1 = 6$

Basic Indices: $B_2 = \{1, 3\}$.

Matrix of Basic Columns and Its Determinant:

$$A_{B_2} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \det A_{B_2} = 0 \tag{6}$$

As $\det A_{B_2} = 0$, the basic feasible solution relative to B_2 does not exist.

Basic Indices: $B_3 = \{1, 4\}$.

Matrix of Basic Columns and Its Determinant:

$$A_{B_3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \det A_{B_3} = 1 \neq 0 \tag{7}$$

Vector of Basic Variables: $\boldsymbol{x}_{B_3} = \begin{pmatrix} x_1 & x_4 \end{pmatrix}^T$.

Set $x_2 = x_3 = 0$ and compute

$$\hat{\boldsymbol{x}}_{B_3} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_4 \end{pmatrix} = A_{B_3}^{-1} b = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \geq \mathbf{0} \tag{8}$$

Basic Feasible Solution Relative to B_1:

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 3 \end{pmatrix} \tag{9}$$

Cost at $\hat{\boldsymbol{x}}_3$: $c^T\hat{\boldsymbol{x}}_1 = 6$

Basic Indices: $B_4 = \{2, 3\}$.

Matrix of Basic Columns and Its Determinant:

$$A_{B_4} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \det A_{B_4} = -1 \neq 0 \tag{10}$$

Vector of Basic Variables: $\boldsymbol{x}_{B_4} = \begin{pmatrix} x_2 & x_3 \end{pmatrix}^T$.

Set $x_1 = x_4 = 0$ and compute

$$\hat{\boldsymbol{x}}_{B_4} = \begin{pmatrix} \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = A_{B_4}^{-1} b = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \geq \mathbf{0} \tag{11}$$

Basic Feasible Solution Relative to B_4:

$$\hat{\boldsymbol{x}}_4 = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix} \tag{12}$$

Cost at $\hat{\boldsymbol{x}}_1$: $c^T\hat{\boldsymbol{x}}_1 = 3$

Basic Indices: $B_5 = \{2, 4\}$.

Matrix of Basic Columns and Its Determinant:

$$A_{B_5} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \det A_{B_5} = 1 \neq 0 \tag{13}$$

Vector of Basic Variables: $\boldsymbol{x}_{B_5} = \begin{pmatrix} x_2 & x_4 \end{pmatrix}^T$.

Set $x_1 = x_3 = 0$ and compute

$$\hat{\boldsymbol{x}}_{B_5} = \begin{pmatrix} \hat{x}_2 \\ \hat{x}_4 \end{pmatrix} = A_{B_5}^{-1} b = \begin{pmatrix} 6 \\ -3 \end{pmatrix} \ngeq \mathbf{0} \tag{14}$$

As $\hat{\boldsymbol{x}}_{B_5} \ngeq \mathbf{0}$, the basic feasible solution relative to B_5 does not exist.

Basic Indices: $B_6 = \{3, 4\}$.

Matrix of Basic Columns and Its Determinant:

$$A_{B_6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \det A_{B_6} = 1 \neq 0 \tag{15}$$

Vector of Basic Variables: $\boldsymbol{x}_{B_6} = \begin{pmatrix} x_3 & x_4 \end{pmatrix}^T$.

Set $x_1 = x_2 = 0$ and compute

$$\hat{\boldsymbol{x}}_{B_6} = \begin{pmatrix} \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = A_{B_6}^{-1} b = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \geq \mathbf{0} \tag{16}$$

Basic Feasible Solution Relative to B_1:

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix} \tag{17}$$

Cost at $\hat{\boldsymbol{x}}_6$: $c^T\hat{\boldsymbol{x}}_6 = 0$

Optimal Cost: $\min\{c^T\hat{\boldsymbol{x}}_1, c^T\hat{\boldsymbol{x}}_3, c^T\hat{\boldsymbol{x}}_4, c^T\hat{\boldsymbol{x}}_6\} = 0$

Optimal Basic Feasible Solution: $\hat{\boldsymbol{x}}_6$

5. (a) The given LP problem is feasible iff it has a basic feasible solution (BFS). $\hat{x}$ is a BFS iff $\hat{x}$ is feasible solution and there exists an integer $k \in \{1, \ldots, n\}$ such that $\hat{x}_j = 0$ for each $j \in \{1, \ldots, n\} \setminus \{k\}$. Hence, $\hat{x}$ is BFS iff there exists an integer $k \in \{1, \ldots, n\}$ such that $\hat{x}_j = 0$ for each $j \in \{1, \ldots, n\} \setminus \{k\}$ and

$$a_k \hat{x}_k = b, \quad \hat{x}_k \geq 0 \tag{18}$$

Since (18) has a solution iff $a_k > 0$, we conclude that $\hat{x}$ is a BFS iff there exists an integer $k \in \{1, \ldots, n\}$ such that $a_k > 0$, $\hat{x}_k = b/a_k$ and $\hat{x}_j = 0$ for each $j \in \{1, \ldots, n\} \setminus \{k\}$. Hence, the given LP is feasible iff $a_k > 0$ for some $k \in \{1, \ldots, n\}$.

(b) If the given LP is feasible, then

$$\mathcal{B} = \left\{ (\hat{x}_1 \cdots \hat{x}_n)^T : a_k > 0, \hat{x}_k = b/a_k, \hat{x}_j = 0 \text{ for } 1 \le k, j \le n, k \neq j \right\}$$

is the set of all basic feasible solutions. Then, the optimal cost value is

$$\min \left\{ \sum_{j=1}^{n} c_j \hat{x}_j : [\hat{x}_1 \cdots \hat{x}_n]^T \in \mathcal{B} \right\} = \min \left\{ \frac{bc_k}{a_k} : a_k > 0, 1 \le k \le n \right\}$$

6. The given LP is feasible iff it has a basic feasible solution (BFS). $\hat{x}$ is a BFS iff $\hat{x}$ satisfies all constraints and there exist integers $k_1, k_2 \in \{1, \ldots, n\}$ such that $k_1 < k_2$ and $\hat{x}_j = 0$ for each $j \in \{1, \ldots, n\} \setminus \{k_1, k_2\}$. Hence, $\hat{x}$ is BFS iff there exist integers $k_1, k_2 \in \{1, \ldots, n\}$ such that $k_1 < k_2$, $\hat{x}_j = 0$ for each $j \in \{1, \ldots, n\} \setminus \{k_1, k_2\}$ and

$$a_{k_1}\hat{x}_{k_1} + a_{k_2}\hat{x}_{k_2} = b \tag{19}$$

$$\hat{x}_{k_1} + \hat{x}_{k_2} = 1 \tag{20}$$

$$x_{k_1} \ge 0, \quad x_{k_2} \ge 0 \tag{21}$$

Since

$$\hat{x}_{k_1} = \frac{a_{k_2} - b}{a_{k_2} - a_{k_1}}, \quad \hat{x}_{k_2} = \frac{b - a_{k_1}}{a_{k_2} - a_{k_1}}$$

is the solution to (19), (20), we conclude that $\hat{x}$ is a BFS iff there exist integers $k_1, k_2 \in \{1, \ldots, n\}$ such that $k_1 < k_2$, $\hat{x}_j = 0$ for each $j \in \{1, \ldots, n\} \setminus \{k_1, k_2\}$ and

$$\hat{x}_{k_1} = \frac{a_{k_2} - b}{a_{k_2} - a_{k_1}} \ge 0, \quad \hat{x}_{k_2} = \frac{b - a_{k_1}}{a_{k_2} - a_{k_1}} \ge 0 \tag{22}$$

Let us show that (22) does not hold when $k_1 < k_2 \le n-1$. When, $k_1 < k_2 \le n-1$, we have $a_{k_1} < a_{k_2} < b$, and consequently, $a_{k_2} - a_{k_1} > 0$, $a_{k_2} - b < 0$, $b - a_{k_1} > 0$. Thus, $\hat{x}_{k_1} < 0$, $\hat{x}_{k_2} > 0$ when $k_1 < k_2 \le n-1$.

Now, we show that (22) holds when $k_1 < k_2 = n$. When $k_1 < k_2 = n$, we have $a_{k_1} < b < a_{k_2}$, and consequently, $a_{k_2} - a_{k_1} > 0$, $a_{k_2} - b > 0$, $b - a_{k_1} > 0$. Hence, $\hat{x}_{k_1} > 0$, $\hat{x}_{k_2} > 0$ when $k_1 < k_2 = n$. Therefore,

$$\mathcal{B} = \left\{ (\hat{x}_1 \cdots \hat{x}_n)^T : \hat{x}_k = \frac{a_n - b}{a_n - a_k}, \hat{x}_n = \frac{b - a_k}{a_n - a_k}, \hat{x}_j = 0 \text{ for } 1 \le k, j \le n-1, k \neq j \right\}$$

is the set of all basic feasible solutions. Since $\mathcal{B} \neq \emptyset$, the problem is feasible. Then, the optimal cost value is

$$\min \left\{ \sum_{j=1}^{n} c_j \hat{x}_j : (\hat{x}_1 \cdots \hat{x}_n)^T \in \mathcal{B} \right\} = \min_{1 \le k \le n-1} \left\{ c_k \frac{a_n - b}{a_n - a_k} + c_n \frac{b - a_k}{a_n - a_k} \right\}$$

7. (a) The dual of the given LP is

$$
\begin{aligned}
&\text{Maximize } 3y_1 + y_2 \\
&\text{Subject to} \\
&\qquad y_1 + y_2 \leq \quad 1 \\
&\qquad 2y_1 + y_2 \leq \ -1 \\
&\qquad y_1 - y_2 \leq \quad 1
\end{aligned}
$$

Applying the graphical method to the dual LP, we obtain the dual optimal solution

$$y_1 = 0, \quad y_2 = -1$$

(see Figure 6 at the end of this document).

In order for x_1, x_2, x_3 to be an optimal solution to the primal, the following equilibrium conditions have to hold:

$$
\begin{aligned}
x_1(y_1 + y_2 - 1) &= 0 \\
x_2(2y_1 + y_2 + 1) &= 0 \\
x_3(y_1 - y_2 - 1) &= 0
\end{aligned}
$$

For $y_1 = 0, y_2 = -1$, the first equilibrium condition yields $x_1 = 0$.

For $x_1 = 0$, the primal equality constraints become

$$
\begin{aligned}
2x_2 + x_3 &= 3 \\
x_2 - x_3 &= 1
\end{aligned}
$$

Solving the above system, we get $x_2 = 4/3$, $x_3 = 1/3$. Hence,

$$x_1 = 0, \quad x_2 = 4/3, \quad x_3 = 1/3$$

is the optimal solution solution to the primal problem (i.e., to the given problem).

(b) The dual of the given LP is

$$
\begin{aligned}
&\text{Maximize } y_1 + 4y_2 \\
&\text{Subject to} \\
&\qquad y_1 + 2y_2 \leq \ -1 \\
&\qquad 2y_1 + y_2 \leq \ -1 \\
&\qquad y_1 - y_2 \leq \ -1
\end{aligned}
$$

The level set $\{(y_1, y_2) : y_1 + 4y_2 = c\}$ has an intersection with the feasible domain for any c (see Figure 7 at the end of the document), and consequently, the optimal dual objective value is not finite. Then, due to the duality principle, the primal is not feasible.

(c) The dual of the given LP is

$$\begin{aligned}
&\text{Maximize } -y_1+y_2\\
&\text{Subject to}\\
&\qquad 2y_1+y_2 \le 2\\
&\qquad y_1+3y_2 \le 3\\
&\qquad -2y_1+y_2 \le 3\\
&\qquad 3y_1+2y_2 \le 6\\
&\qquad -2y_1+y_2 \le 4
\end{aligned}$$

Applying the graphical method to the dual LP, we obtain the dual optimal solution

$$y_1=-6/7, \quad y_2=9/7$$

(see Figure 8 at the end of this document).

In order for x_1, x_2, x_3, x_4, x_5 to be an optimal solution to the primal, the following equilibrium conditions have to hold:

$$x_1(2y_1+y_2-2)=0 \tag{23}$$
$$x_2(y_1+3y_2-1)=0 \tag{24}$$
$$x_3(-2y_1+y_2-3)=0 \tag{25}$$
$$x_4(3y_1+2y_2-6)=0 \tag{26}$$
$$x_5(-2y_1-y_2-4)=0 \tag{27}$$

For $y_1=-6/7, y_2=9/7$, equilibrium conditions (23), (26), (27) yield

$$x_1=0, \quad x_4=0, \quad x_5=0.$$

For $x_1=0$, $x_4=0$, $x_5=0$, the primal equality constraints become

$$\begin{aligned}
x_2-2x_3&=-1\\
3x_2+x_3&=1
\end{aligned}$$

Solving the above system, we get $x_2=1/7$, $x_3=4/7$. Hence,

$$x_1=0, \quad x_2=1/7, \quad x_3=4/7, \quad x_4=0, \quad x_5=0$$

is the optimal solution solution to the primal problem (i.e., to the given problem).

(d) The dual of the given LP is

$$\begin{aligned}
&\text{Maximize } 4y_1+3y_2\\
&\text{Subject to}\\
&\qquad y_1+2y_2 \le 2\\
&\qquad y_1-2y_2 \le 3\\
&\qquad 2y_1+3y_2 \le 5\\
&\qquad y_1+y_2 \le 2\\
&\qquad 3y_1+y_2 \le 3\\
&\qquad y_1 \ge 0, y_2 \ge 0
\end{aligned}$$

Applying the graphical method to the dual LP, we obtain the dual optimal solution

$$y_1 = 4/5, \quad y_2 = 3/5$$

(see Figure 9 at the end of this document).

In order for x_1, x_2, x_3, x_4, x_5 to be an optimal solution to the primal, the following equilibrium conditions have to hold:

$$x_1(y_1 + 2y_2 - 2) = 0 \tag{28}$$
$$x_2(y_1 - 2y_2 - 3) = 0 \tag{29}$$
$$x_3(2y_1 + 3y_2 - 5) = 0 \tag{30}$$
$$x_4(y_1 + y_2 - 2) = 0 \tag{31}$$
$$x_5(3y_1 + y_2 - 3) = 0 \tag{32}$$
$$y_1(x_1 + x_2 + 2x_3 + x_4 + 3x_5 - 4) = 0 \tag{33}$$
$$y_2(2x_1 - 2x_2 + 3x_3 + x_4 + x_5 - 3) = 0 \tag{34}$$

For $y_1 = 4/5, y_2 = 3/5$, equilibrium conditions (29) – (31) yield

$$x_2 = 0, \quad x_3 = 0, \quad x_4 = 0.$$

For $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $y_1 = 4/5, y_2 = 3/5$, equilibrium conditions (33), (34) imply For $x_1 = 0$, $x_4 = 0$, $x_5 = 0$, the primal equality constraints become

$$\begin{aligned} x_1 + 3x_5 &= 4 \\ 2x_1 + x_5 &= 3 \end{aligned}$$

Solving the above system, we get $x_1 = 1$, $x_5 = 1$. Hence,

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 1$$

is the optimal solution solution to the primal problem (i.e., to the given problem).

8. (a) The dual of the given LP is

$$\begin{aligned} &\text{Maximize } 16y_2 \\ &\text{Subject to} \\ &\quad 5y_1 + y_2 \le 1 \\ &\quad -6y_1 - y_2 \le 5 \\ &\quad 4y_1 + 6y_2 \le 2 \\ &\quad -2y_1 + 9y_2 \le 13 \end{aligned}$$

In order for x_1, x_2, x_3, x_4 to be an optimal primal solution, the following equilibrium conditions have to hold:

$$x_1(5y_1 + y_2 - 1) = 0 \tag{35}$$
$$x_2(-6y_1 - y_2 - 5) = 0 \tag{36}$$
$$x_3(4y_1 + 6y_2 - 2) = 0 \tag{37}$$
$$x_4(-2y_1 + 9y_2 - 13) = 0 \tag{38}$$

For $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$, equilibrium condition (36), (37) reduce to the following system:

$$\begin{aligned} 6y_1 + y_2 &= -5 \\ 4y_1 + 6y_2 &= 2 \end{aligned}$$

Solution to the above system is $y_1 = -1$, $y_2 = 1$. It is straightforward to show that $y_1 = -1$, $y_2 = 1$ is a dual feasible solution. Since $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$ and $y_1 = -1$, $y_2 = 1$ satisfy the equilibrium conditions, we conclude that $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, $x_4 = 0$ is an optimal primal solution.

(b) The dual of the given LP is

$$\begin{array}{l} \text{Maximize } 24y_1 + 20y_2 + 8y_3 \\ \text{Subject to} \\ \quad 4y_1 + y_2 \quad +2y_3 \le -6 \\ \quad 6y_1 + 4y_2 \quad +3y_3 \le -6 \\ \quad 2y_1 + 3y_2 \quad +y_3 \le -4 \\ \quad y_1 \qquad\qquad\quad \le 0 \end{array}$$

In order for x_1, x_2, x_3, x_4 to be an optimal primal solution, the following equilibrium conditions have to hold:

$$x_1(4y_1 + 4y_2 + 2y_3 + 6) = 0 \tag{39}$$

$$x_2(6y_1 + 4y_2 + 3y_3 + 6) = 0 \tag{40}$$

$$x_3(2y_1 + 3y_2 + y_3 + 4) = 0 \tag{41}$$

$$x_4 y_1 = 0 \tag{42}$$

For $x_1 = 2$, $x_2 = 0$, $x_3 = 4$, $x_4 = 8$, equilibrium condition (39), (41), (42) reduce to the following system:

$$\begin{aligned} 4y_1 + 4y_2 + 2y_3 &= -6 \\ 2y_1 + 3y_2 + y_3 &= -4 \\ y_1 &= 0 \end{aligned}$$

Solution to the above system is $y_1 = 0$, $y_2 = -1$, $y_3 = -1$. It is straightforward to show that $y_1 = 0$, $y_2 = -1$, $y_3 = -1$ is a dual feasible solution. Since $x_1 = 2$, $x_2 = 0$, $x_3 = 4$, $x_4 = 8$ and $y_1 = 0$, $y_2 = -1$, $y_3 = -1$ satisfy the equilibrium conditions, we conclude that $x_1 = 2$, $x_2 = 0$, $x_3 = 4$, $x_4 = 8$ is an optimal primal solution.

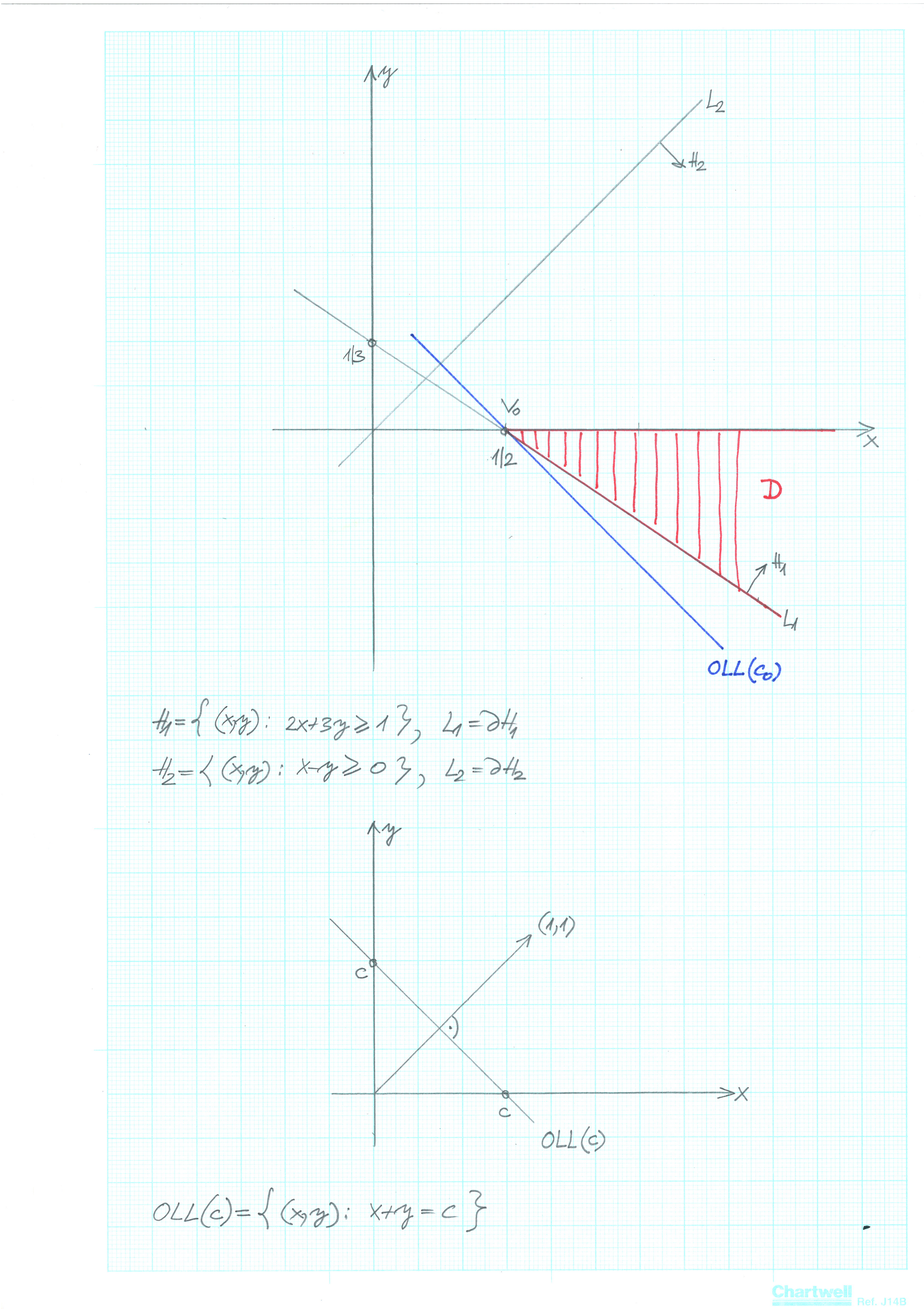

Figure 1: Question 3a

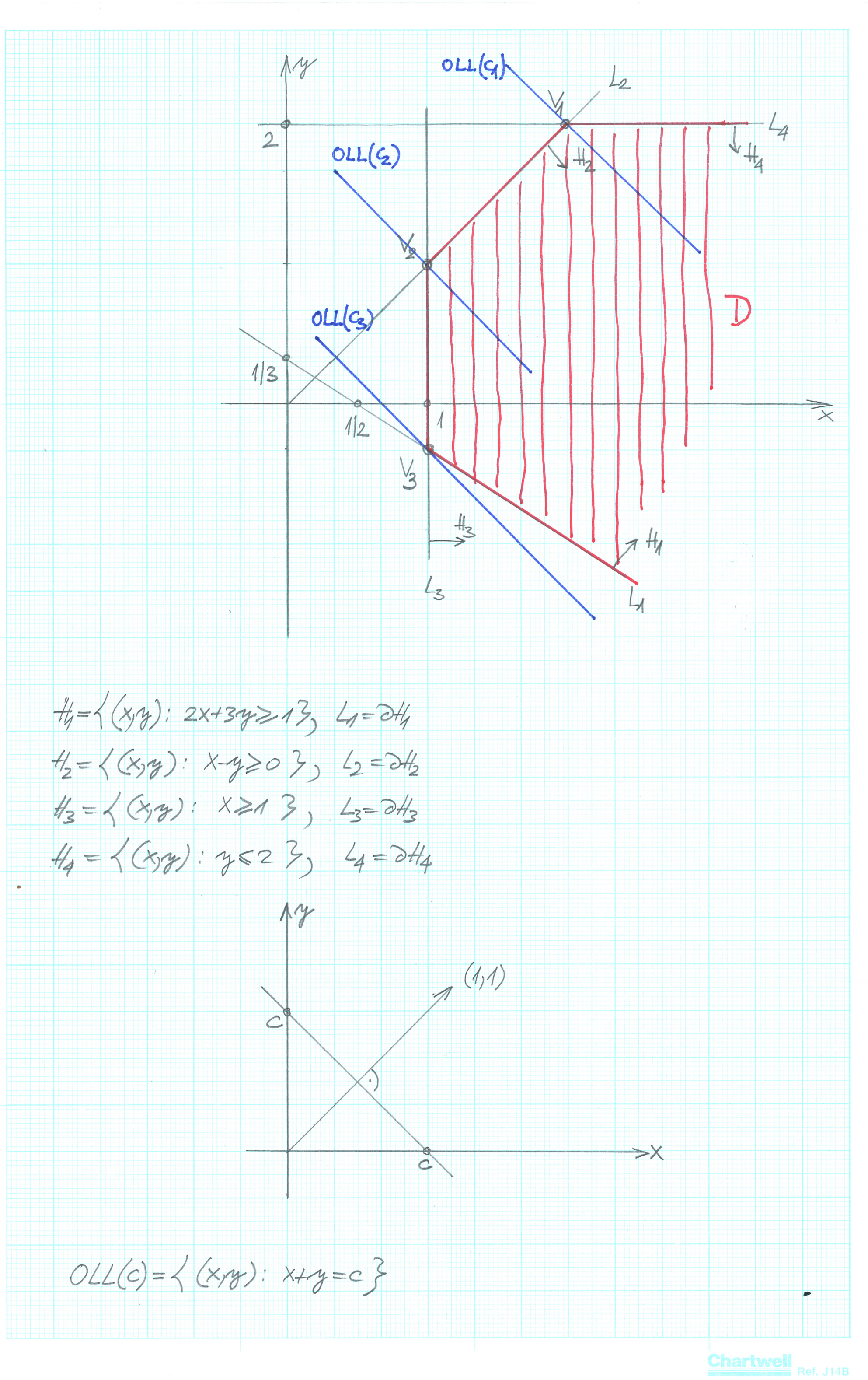

Figure 2: Question 3b

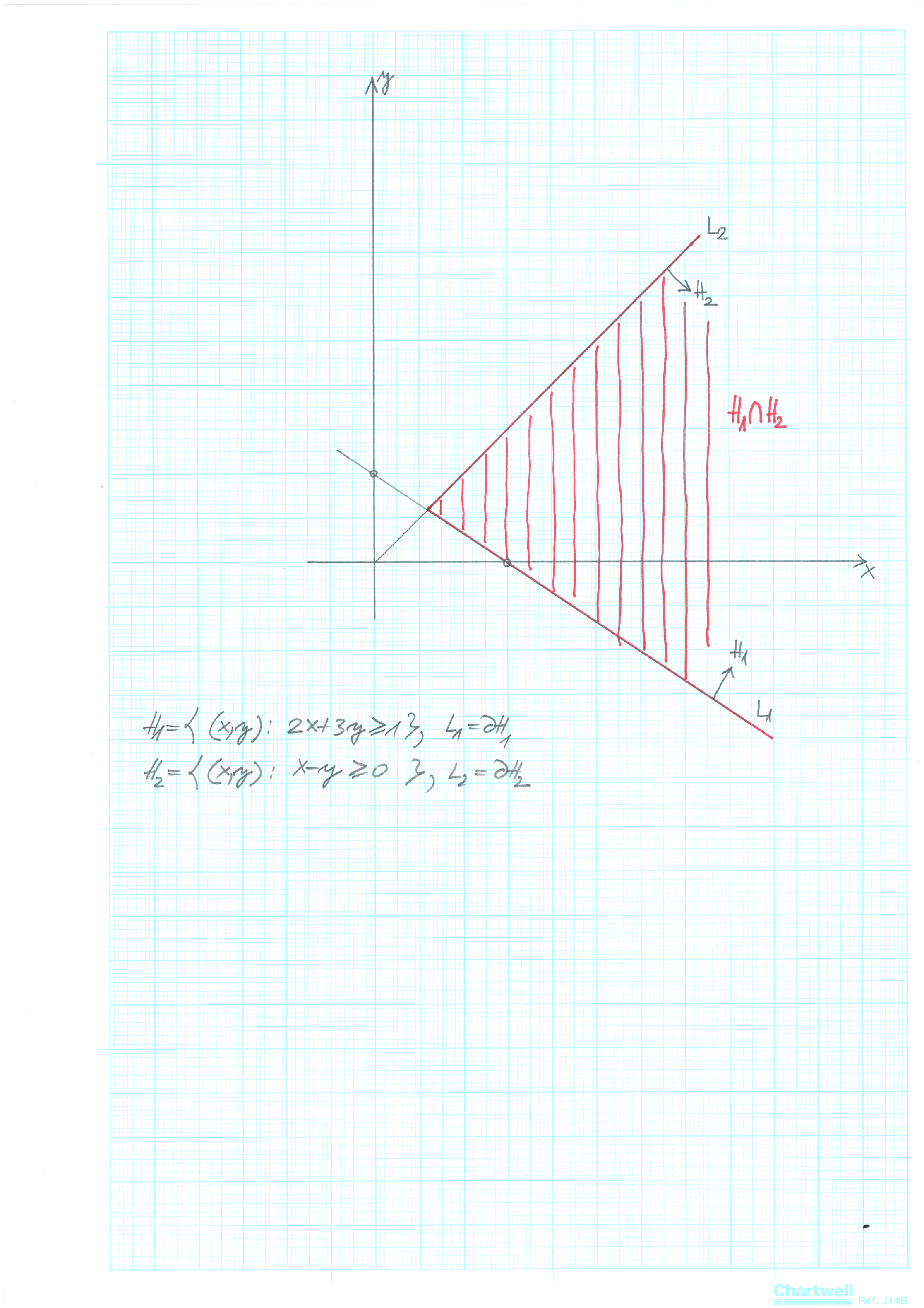

Figure 3: Question 3c

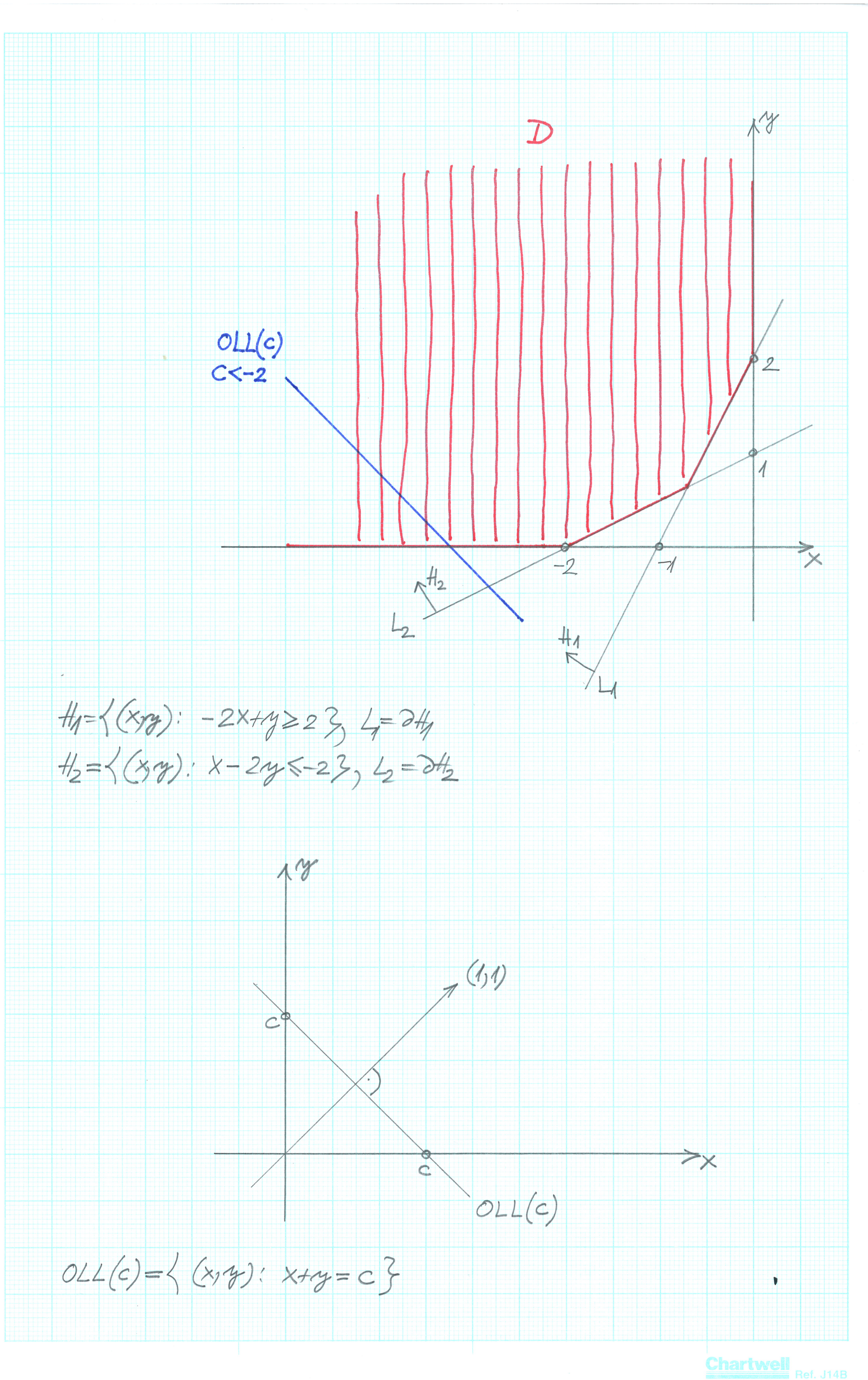

Figure 4: Question 3d

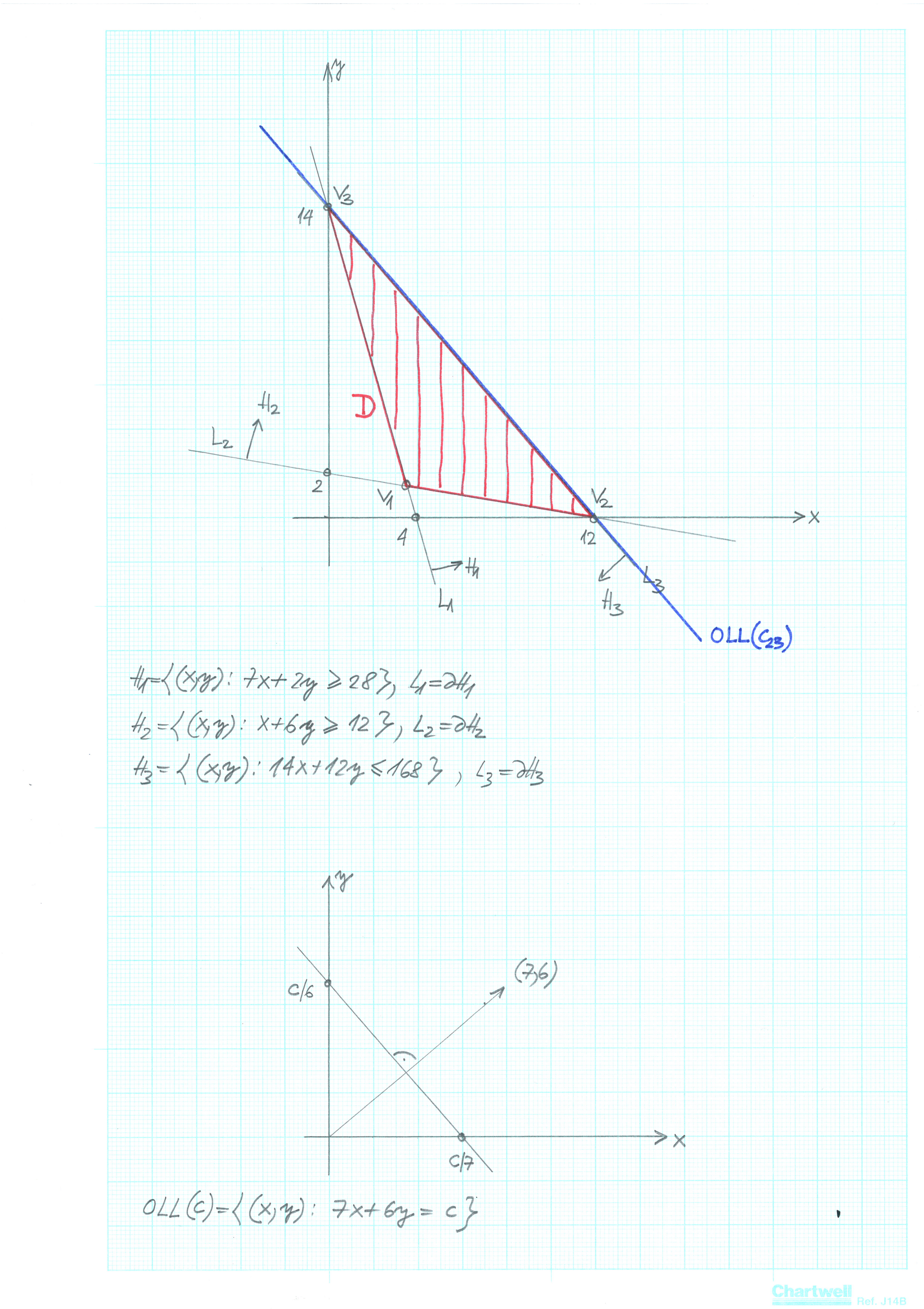

Figure 5: Question 3e

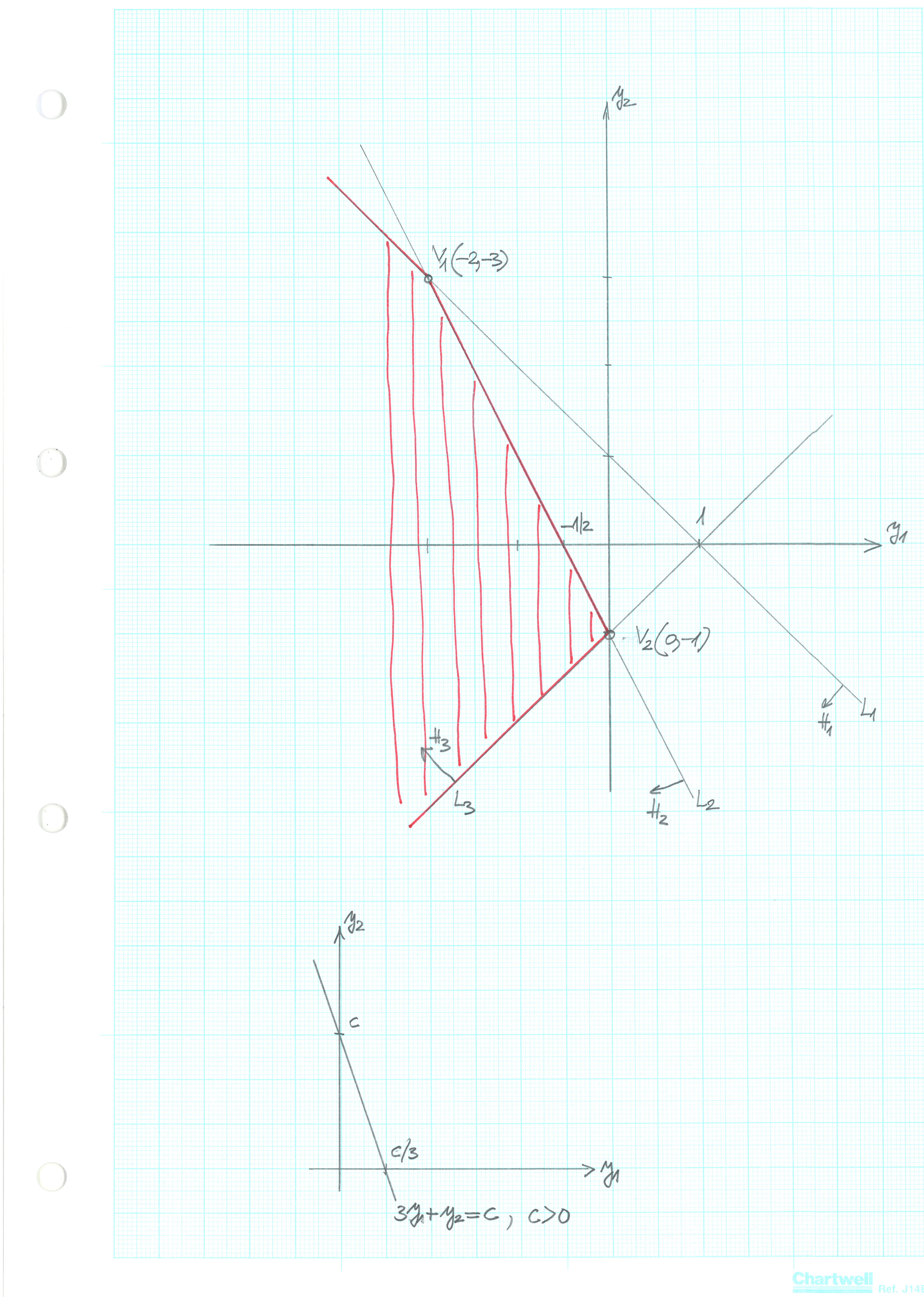

Figure 6: Question 7a

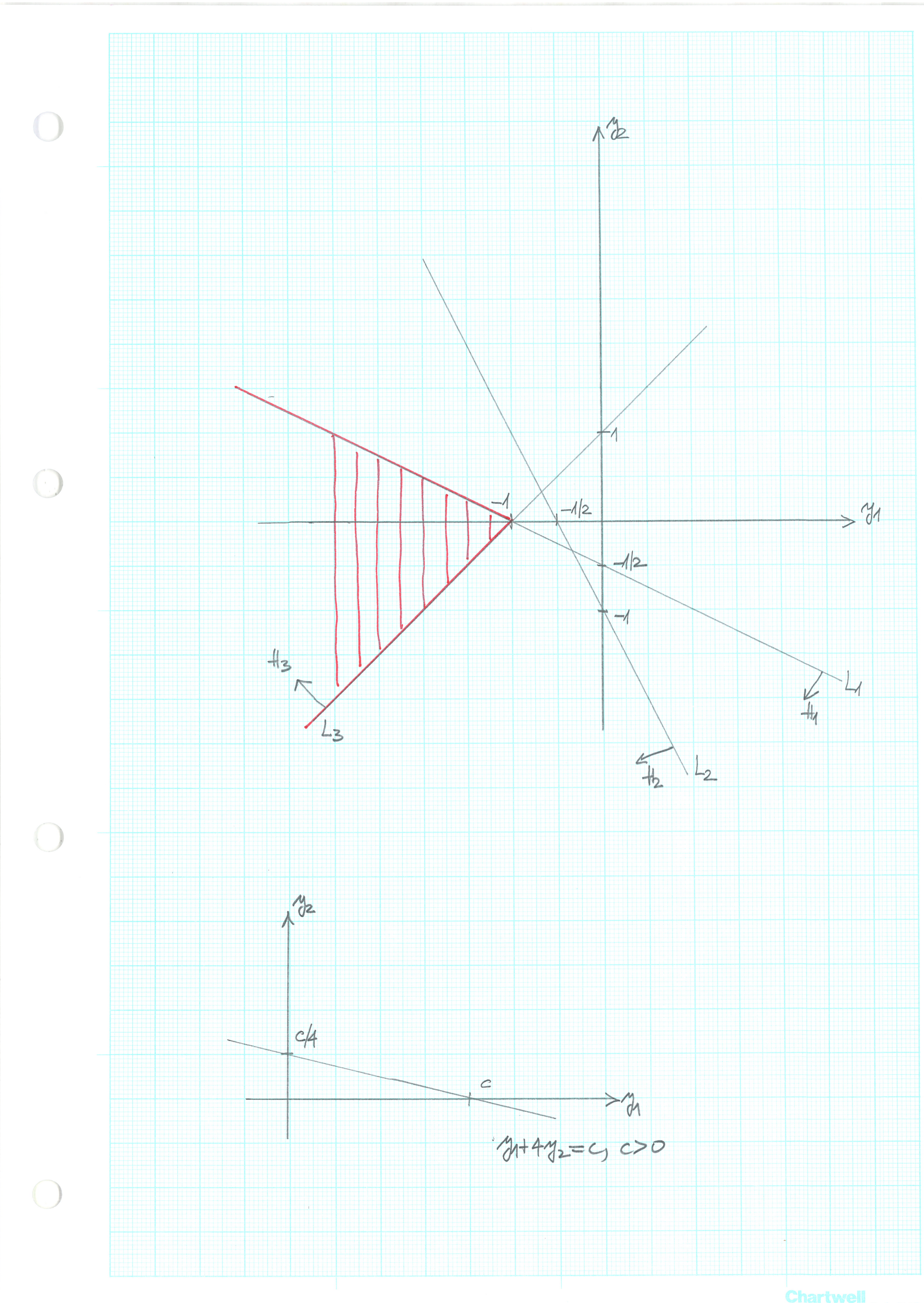

Figure 7: Question 7b

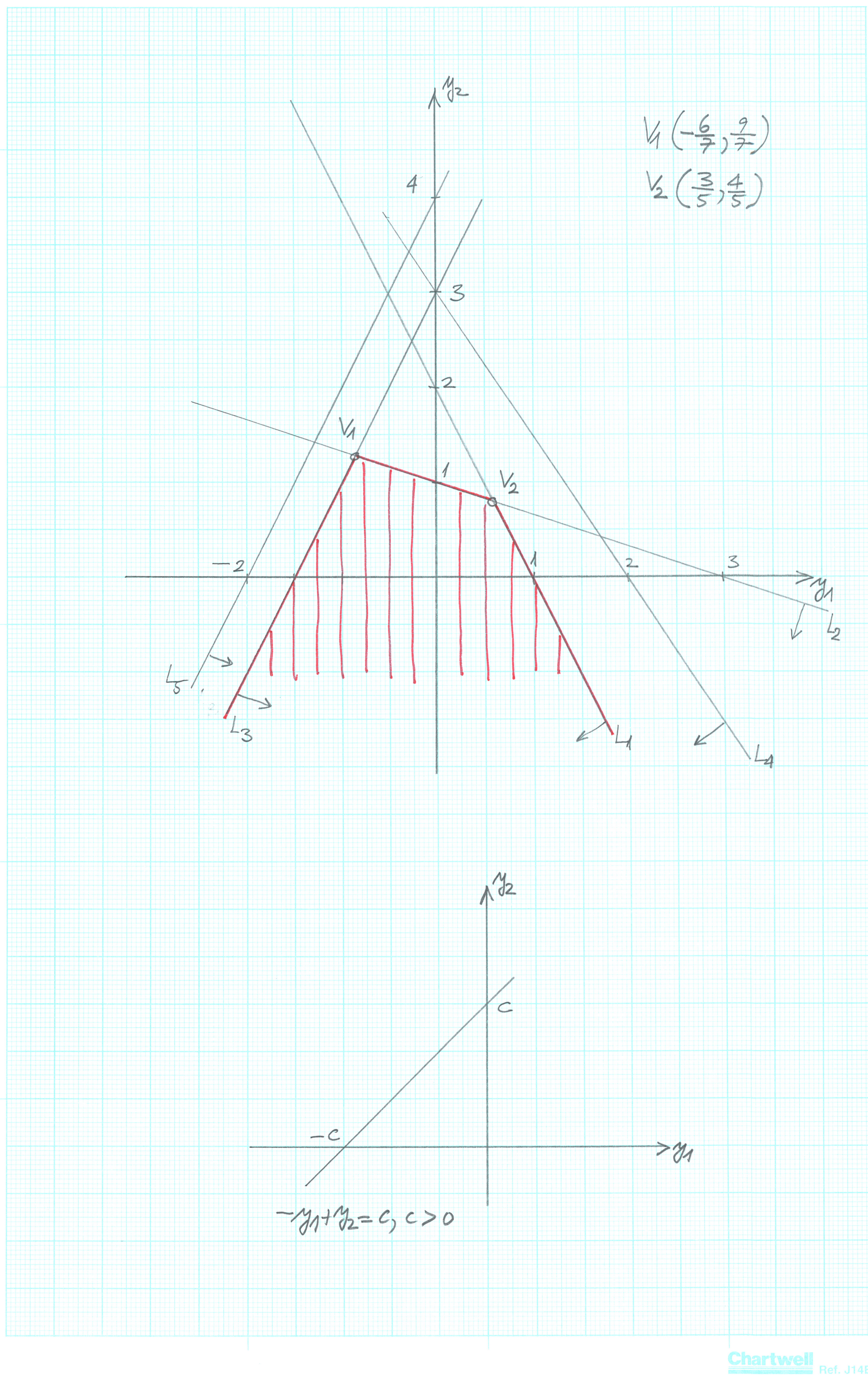

Figure 8: Question 7c

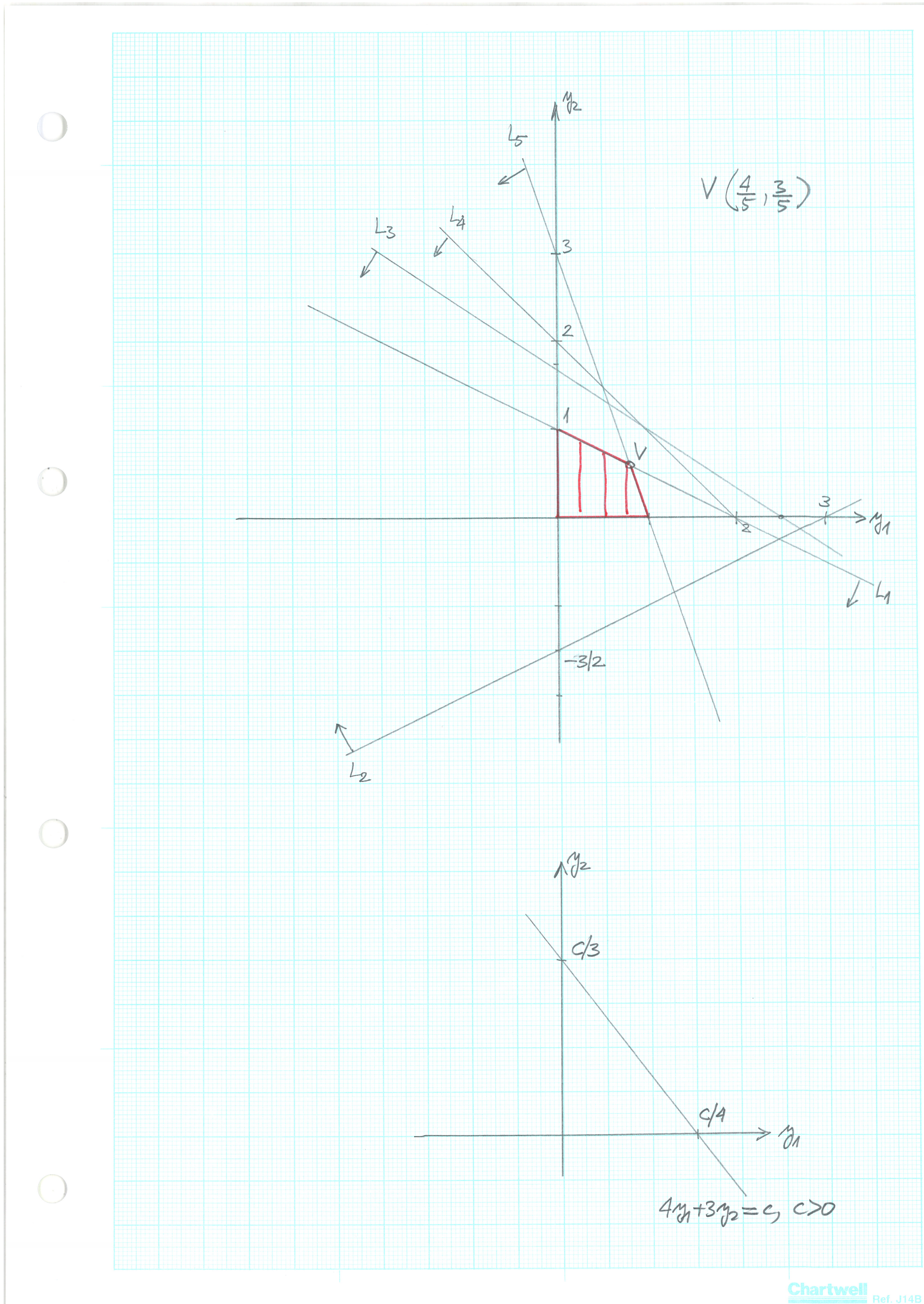

Figure 9: Question 7d