

Stochastic Optimisation

Problem Sheet 1

1. Let X_1 and X_2 be independent Poisson random variables with means λ_1 and λ_2 . Using generating functions, show that $X_1 + X_2$ is Poisson with mean $\lambda_1 + \lambda_2$.

2. Let T have an Exponential distribution with parameter μ .

(a) Show that $\mathbb{E}[T] = 1/\mu$ and that μT has an Exponential distribution with parameter 1.
Hint. The first part can be done using generating functions or integration by parts. For the second part, use the expression for the cdf of an exponential distribution, or use the formula for transformations of random variables, or use generating functions.

(b) Show that the distribution of T is “memoryless”, i.e., for all $t, u > 0$,

$$P(T > t + u | T > u) = P(T > t).$$

3. Let X_1, X_2, \dots be iid Bernoulli random variables with parameter $p \in [0, 1]$, denoted $\text{Bern}(p)$. Suppose $q \in [0, 1]$ is bigger than p . Show that

$$\mathbb{P}(X_1 + X_2 + \dots + X_n > nq) \leq \exp(-nK(q; p)) \text{ where } K(q; p) = q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p},$$

with $x \log x$ defined to be 0 if $x = 0$.

4. Let X_1, X_2, \dots be iid Poisson random variables with parameter $\lambda \in \mathbb{R}_+$, denoted $\text{Poi}(\lambda)$. Suppose $\mu \in \mathbb{R}_+$ is smaller than λ . Show that

$$\mathbb{P}(X_1 + X_2 + \dots + X_n < n\mu) \leq \exp(-nI(\mu; \lambda)) \text{ where } I(\mu; \lambda) = \mu \log \frac{\mu}{\lambda} - \mu + \lambda.$$

5. Let X_1, X_2, \dots be iid Exponential random variables with parameter $\lambda \in \mathbb{R}_+$, denoted $\text{Exp}(\lambda)$. Suppose $x \in \mathbb{R}_+$ is bigger than $1/\lambda$. Show that

$$\mathbb{P}(X_1 + X_2 + \dots + X_n > nx) \leq \exp(-nJ(x; \lambda)) \text{ where } J(x; \lambda) = \lambda x - 1 - \log(\lambda x).$$

6. (a) Let Z be normally distributed with mean zero and unit variance, written $Z \sim N(0, 1)$. Recall that Z has density

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Show that its moment generating function (mgf) is given by $\mathbb{E}[e^{\theta Z}] = e^{\theta^2/2}$.

- (b) Let X_1, X_2, \dots be iid $N(\mu, \sigma^2)$, i.e., normally distributed with mean μ and variance σ^2 . Let γ be bigger than μ . Show that

$$\mathbb{P}(X_1 + X_2 + \dots + X_n > n\gamma) \leq \exp\left(-n \frac{(\gamma - \mu)^2}{2\sigma^2}\right).$$

7. Let \mathbf{X} be a random variable taking values in \mathbb{R}^d . Its mgf as defined as the function from \mathbb{R}^d to \mathbb{R} given by

$$M_{\mathbf{X}}(\boldsymbol{\theta}) = \mathbb{E}[\exp(\langle \boldsymbol{\theta}, \mathbf{X} \rangle)] = \mathbb{E}[\exp(\boldsymbol{\theta}^T \mathbf{X})], \quad \boldsymbol{\theta} \in \mathbb{R}^d,$$

where $\langle \boldsymbol{\theta}, \mathbf{X} \rangle$ denotes the inner product of $\boldsymbol{\theta}$ and \mathbf{X} .

Let $H(\boldsymbol{\theta}, y)$ denote the half-space in \mathbb{R}^d given by

$$H(\boldsymbol{\theta}, y) = \{\mathbf{x} \in \mathbb{R}^d : \langle \boldsymbol{\theta}, \mathbf{x} \rangle \geq y\}.$$

Show that

$$\mathbb{P}(\mathbf{X} \in H(\boldsymbol{\theta}, y)) \leq e^{-\eta y} M_{\mathbf{X}}(\eta \boldsymbol{\theta}), \quad \forall \eta > 0.$$

8. Let X_1, X_2, \dots be iid random variables with mean μ and taking values in the interval $[a, b]$, where a and b are finite. Show that

$$\mathbb{P}\left(\sum_{i=1}^n X_i - n\mu > nt\right) \leq \exp\left(-\frac{2nt^2}{(b-a)^2}\right).$$