Stochastic Optimisation

Problem Sheet 1

$**$ **Please hand in solutions to question 8 on this sheet.** $**$

1. Let X_1 and X_2 be independent Poisson random variables with means λ_1 and λ_2. Using generating functions, show that $X_1 + X_2$ is Poisson with mean $\lambda_1 + \lambda_2$.

2. Let T have an Exponential distribution with parameter μ.

   (a) Show that $\mathbb{E}[T] = 1/\mu$ and that μT has an Exponential distribution with parameter 1. *Hint.* The first part can be done using generating functions or integration by parts. For the second part, use the expression for the cdf of an exponential distribution, or use the formula for transformations of random variables, or use generating functions.

   (b) Show that the distribution of T is "memoryless", i.e., for all $t, u > 0$,

   $$P(T > t + u | T > u) = P(T > t).$$

3. Let $X_1, X_2, \ldots$ be iid Bernoulli random variables with parameter $p \in [0, 1]$, denoted Bern(p). Suppose $q \in [0, 1]$ is bigger than p. Show that

   $$\mathbb{P}(X_1 + X_2 + \ldots + X_n > nq) \leq \exp(-nK(q; p)) \text{ where } K(q; p) = q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p},$$

   with $x \log x$ defined to be 0 if $x = 0$.

4. Let $X_1, X_2, \ldots$ be iid Poisson random variables with parameter $\lambda \in \mathbb{R}_+$, denoted Poi(λ). Suppose $\mu \in \mathbb{R}_+$ is smaller than λ. Show that

   $$\mathbb{P}(X_1 + X_2 + \ldots + X_n < n\mu) \leq \exp(-nI(\mu; \lambda)) \text{ where } I(\mu; \lambda) = \mu \log \frac{\mu}{\lambda} - \mu + \lambda.$$

5. Let $X_1, X_2, \ldots$ be iid Exponential random variables with parameter $\lambda \in \mathbb{R}_+$, denoted Exp(λ). Suppose $x \in \mathbb{R}_+$ is bigger than $1/\lambda$. Show that

   $$\mathbb{P}(X_1 + X_2 + \ldots + X_n > nx) \leq \exp(-nJ(x; \lambda)) \text{ where } J(x; \lambda) = \lambda x - 1 - \log(\lambda x).$$

6. (a) Let Z be normally distributed with mean zero and unit variance, written $Z \sim N(0, 1)$. Recall that Z has density

   $$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

   Show that its moment generating function (mgf) is given by $\mathbb{E}[e^{\theta Z}] = e^{\theta^2/2}$.

(b) Let $X_1, X_2, \ldots$ be iid $N(\mu, \sigma^2)$, i.e., normally distributed with mean μ and variance σ^2. Let γ be bigger than μ. Show that

$$\mathbb{P}(X_1 + X_2 + \ldots + X_n > n\gamma) \leq \exp\Big(-n\frac{(\gamma - \mu)^2}{2\sigma^2}\Big).$$

7. Let $\mathbf{X}$ be a random variable taking values in $\mathbb{R}^d$. Its mgf as defined as the function from $\mathbb{R}^d$ to $\mathbb{R}$ given by

$$M_{\mathbf{X}}(\boldsymbol{\theta}) = \mathbb{E}[\exp(\langle \boldsymbol{\theta}, \mathbf{X} \rangle)] = \mathbb{E}[\exp(\boldsymbol{\theta}^T \mathbf{X})], \quad \boldsymbol{\theta} \in \mathbb{R}^d,$$

where $\langle \boldsymbol{\theta}, \mathbf{X} \rangle$ denotes the inner product of $\boldsymbol{\theta}$ and $\mathbf{X}$.

Let $H(\boldsymbol{\theta}, y)$ denote the half-space in $\mathbb{R}^d$ given by

$$H(\boldsymbol{\theta}, y) = \{\mathbf{x} \in \mathbb{R}^d : \langle \boldsymbol{\theta}, \mathbf{x} \rangle \geq y\}.$$

Show that

$$\mathbb{P}(\mathbf{X} \in H(\boldsymbol{\theta}, y)) \leq e^{-\eta y} M_{\mathbf{X}}(\eta \boldsymbol{\theta}), \quad \forall\, \eta > 0.$$

8. Let $X_1, X_2, \ldots$ be iid random variables with mean μ and taking values in the interval $[a, b]$, where a and b are finite. Show that

$$\mathbb{P}\Big(\sum_{i=1}^{n} X_i - n\mu > nt\Big) \leq \exp\Big(-\frac{2nt^2}{(b-a)^2}\Big).$$