Differential equations

Terminology A differential equation is of the form

$$F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = 0, \tag{1}$$

where F is a function of the *independent variable* x, the *dependent variable* $y(x)$ and derivatives of the dependent variable and n is a positive integer. Expression (1) is a n^{th} order differential equation. The aim is calculate the unknown function $y(x)$.
A **linear** differential equation is one in which the dependent variable and its derivatives only appear as additive combinations of their first powers. It may be written as

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = b(x). \tag{2}$$

If the equation is not linear then we call it **nonlinear**.
An **initial value problem** for an n^{th} order differential equation means find the solution of (1) on an interval I subject to conditions

$$y(x_0) = y_0,\ \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = y_1 \ldots \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}(x_0) = y_{n-1},$$

where $x_o \in I$ and y_0, y_1,...,y_{n-1} are given constants.
A **boundary value problem** applies conditions at different values of the independent variable.

1 First order differential equations

1.1 Direct integration

If $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ subject to $y(b) = y_0$ then

$$y(x) = \int_b^x g(s)\ \mathrm{d}s + y_0. \tag{3}$$

This expression gives the solution irrespective of whether the solution can be evaluated using analytical techniques.

1.2 Separation of variables

Suppose the differential equation is of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(y)f(x)$ then

$$\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int f(x)\ \mathrm{d}x, \tag{4}$$

Then by change of variables we can write

$$\int \frac{1}{g(y)}\ \mathrm{d}y = \int f(x)\ \mathrm{d}x. \tag{5}$$

1.3 Integrating factor

A general linear first-order differential equation takes the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x). \tag{6}$$

The integrating factor is $p(x) = \exp\left(\int^x g(x)\ \mathrm{d}x\right)$ and we multiply (6) by $p(x)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yp(x)\right) = h(x)p(x), \tag{7}$$

which can then be integrated directly to obtain

$$y(x) = \frac{1}{p(x)}\int^x h(s)p(s)\ \mathrm{d}s + \frac{c}{p(x)}, \tag{8}$$

where c is a constant of integration.
Note that it is not necessary to provide a lower limit in the integral that defines the integrating factor. This would merely multiply each term of the differential equation by a constant.

1.4 Homogeneous differential equations

Suppose a first-order differential equation takes the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right). \tag{9}$$

The problem is then said to be **homogeneous**. This may be integrated by substituting $y(x) = v(x)x$ and so $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$. Then (9) becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v), \tag{10}$$

which can be integrated by separating variables

$$\int \frac{1}{f(v) - v}\ \mathrm{d}v = \int \frac{1}{x}\ \mathrm{d}x. \tag{11}$$

This technique often lead to implicit solutions for the dependent variable $y(x)$.

1.5 Uniqueness, existence and domain of validity

Given an initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) \qquad x(t_0) = x_0, \tag{12}$$

- Is there always a solution? *'Existence'*
- Is the solution unique? *'Uniqueness'*
- For what range of t is the solution defined? *'Domain of validity'*

1.5.1 Example: *Existence*

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1+x^2}{x}$, which may be integrated to give $y = \log|x| + \frac{1}{2}x^2 + c$.
(i) If $y(1) = 0$ then $c = 1/2$.
(ii) If $y(0) = 0$ then no value of the constant c can be found to satisfy the condition so the solution does not exist. We might have anticipated difficulties because dy/dx is not defined at $x = 0$.

1.5.2 Example: *Uniqueness*

Consider the differential equation $\dfrac{dx}{dt} = 3x^{2/3}$ subject to $x(0) = 0$.
A trivial solution is that $x(t) = 0$. Alternatively by separation of variables we find that $x(t) = (t+c)^3$ and imposing the condition gives $x(t) = t^3$.
Infact the equation and conditions are also satisfied by the general function

$$x(t) = \begin{cases} (t-a)^3, & t > a, \\ 0, & a > t > b, \\ (t-b)^3, & b > t \end{cases},$$

where $a > 0$ and $b < 0$. Thus the solution is not unique.

1.5.3 Uniqueness and existence theorem: *Non-examinable*

Definition: For $f(x,t)$, the partial derivative is $\dfrac{\partial f}{\partial x} = \lim_{h\to 0} \dfrac{f(x+h,t) - f(x,t)}{h}$.

If the functions $f(x,t)$ and $\partial f/\partial x$ are continuous in the domain $D = \{\alpha < t < \beta, \gamma < x < \delta\}$ such that $(x_0, t_0) \in D$, then there exists a unique solution to the initial value problem

$$\frac{dx}{dt} = f(x,t) \text{ with } x(t_0) = x_0 \text{ for } t_0 - \Delta < t < t_0 + \Delta,$$

where $\alpha < t_0 \pm \Delta < \beta$.

Under the weaker condition that only $f(x,t)$ is continuous then the theorem asserts existence but not uniqueness.

Note that the theorem only guarantees existence/uniqueness in a region close to (x_0, t_0). Also note that if the conditions do not hold, then the theorem does not assert non-existence.

1.5.4 Example: *Domain of validity*

The differential equation $\dfrac{dy}{dx} = 2xy^2$ gives the general solution $y = -\dfrac{1}{x^2+c}$.
Subject to condition $y(0) = -1$, $y(x) = \dfrac{-1}{x^2+1}$. The solution is valid for all values of x.
Subject to condition $y(0) = 1$, $y(x) = \dfrac{1}{1-x^2}$. The solution is valid for $|x| < 1$.

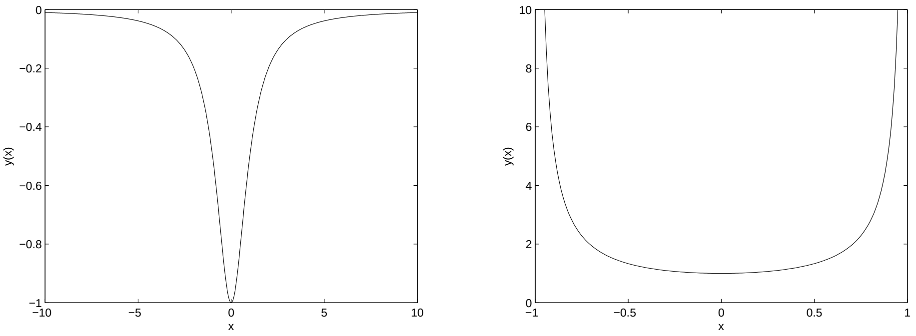

Figure 1: Example 1.5.4: (a) $y(x) = -1/(x^2+1)$ and (b) $y(x) = 1/(1-x^2)$.

1.6 Numerical approximation of solutions: *Euler's method*

Often the solution to an initial value problem is not available by analytical techniques, even though we know the solution exists. In such situations we have to evaluate the solution using numerical methods.

The simplest method is **Euler's method**. Consider the initial value problem $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \qquad y(x_0) = y_0$. The initial tangent to the solution is

$$y = y_0 + f(x_0, y_0)(x - x_0).$$

We follow this tangent upto a point (x_1, y_1), and then examine another tangent

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1).$$

Really this is the local tangent approximation to the initial value problem $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \qquad y(x_1) = y_1$. This problem (& so the solution) are different from the original initial value problem. However as long as $x_1 - x_0$ is relatively small, it may be expected that the solutions are relatively close. This suggests the following iterative update rule, which is known as Euler's method

$$y_{n+1} = y_n + f(x_n, y_n)(x_{n+1} - x_n).$$

Euler's method can work well in some situations - but fails in others where a more elaborate numerical algorithm is required.

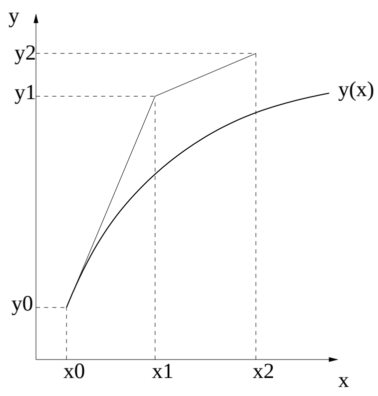

Figure 2: Sketch showing the construction of Euler's method.