2 Second order linear differential equations

In this chapter we focus on *linear* differential equations of the following form

$$L[y] \equiv a(x)\frac{\mathrm{d}y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = f(x).$$
(1)

We define the homogeneous problem to be L[y] = 0 and the inhomogeneous problem to be L[y] = f(x). An important special case is second order linear differential equations with constant coefficients (a, b, c constants) - in this case there is a general method for solution that can be generalised to higher orders.

2.1 Homogeneous equations with constant coefficients

$$L[y] \equiv a\frac{\mathrm{d}y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$
⁽²⁾

2.1.1 Linearity and superposition

Suppose $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation then $\alpha y_1 + \beta y_2$ (α, β constants) is also a solution. This is readily shown:

$$L[\alpha y_1 + \beta y_2] = L[\alpha y_1] + L[\beta y_2] = \alpha L[y_1] + \beta L[y_2] = 0.$$
(3)

2.1.2 Constructing solutions

We look for solutions of the form $y(x) = e^{mx}$, where m is to be determined. Then $dy/dx = me^{mx}$ and $d^2y/dx^2 = m^2 e^{mx}$. Thus (2) is satisfied if

$$\left(am^2 + bm + c\right)e^{mx} = 0. \tag{4}$$

Therefore m satisfies a quadratic equation with solutions

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
(5)

In general this will have two solutions, $\{m_1, m_2\}$ and so there are two solutions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. Then the general solution is given by

$$y = Ae^{m_1 x} + Be^{m_2 x}$$
, with A, B constants. (6)

2.1.3 Examples

(i) $\frac{d^2y}{dx^2} - 4y = 0$. Look for solutions of the form $y(x) = e^{mx}$ and so $m^2 - 4 = 0$. Thus $m = \pm 2$ and the general solution is

$$y(x) = A\mathrm{e}^{2x} + B\mathrm{e}^{-2x}.$$

(ii) $\frac{d^2y}{dx^2} + y = 0$. Look for solutions of the form $y(x) = e^{mx}$ and so $m^2 + 1 = 0$. Thus $m = \pm i$ and the general solution is

$$y(x) = Ae^{ix} + Be^{-ix} = C\cos x + D\sin x$$

(ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$. Look for solutions of the form $y(x) = e^{mx}$ and so $m^2 - 2m + 5 = 0$. Thus $m = 1 \pm 2i$ and the general solution is

$$y(x) = Ae^{(1+2i)x} + B^{(1-2i)x} = e^x \left(C\cos 2x + D\sin 2x\right).$$

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2.1.4 Repeated root

How do we find the general solution when there is only one root to (4)? This occurs if $b^2 = 4ac$ and then m = -b/(2a). In this situation we write $y(x) = v(x)e^{-bx/(2a)}$ and substitute into (2).

$$L[v(x)e^{-bx/(2a)}] = e^{-bx/(2a)}\frac{d^2v}{dx^2} = 0.$$
(7)

Thus v(x) = Ax + B for constant A and B and so the general solution is

$$y(x) = e^{-bx/(2a)} (Ax + B).$$
 (8)

Example: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ leads to characteristic equation $m^2 = 4m + 4 = (m+2)^2 = 0$. Thus the general solution is

$$y(x) = e^{-2x} (Ax + B).$$
 (9)

2.1.5 Initial value problems and boundary value problems

To determine the solution to a particular problem, we need conditions to determine the unknown constants in the general solution. For second order linear differential equations, we generally need two conditions to determine the two constants.

(i) Initial value problems: There are two conditions that the solution must satisfy at x = a. For example we might demand that $y(a) = y_0$ and $dy/dx(a) = d_0$, where y_0 and d_0 are known values. Example: $\frac{d^2y}{dx^2} + \omega^2 y = 0$ subject to y(0) = 1 and y'(0) = 0. This has general solution $y(x) = A \cos \omega x + B \sin \omega x$. Then applying the conditions y(0) = 1 implies A = 1 and y'(0) = 0 implies B = 0 and so

$$y(x) = \cos \omega x.$$

(ii) Boundary value problems: These are conditions that the solution must satisfy at different locations, say x = a and x = b. For example, we might demand y(a) = 0 and y(b) = 0. Boundary value problems do not always have a solution.

Example: Waves on a string. The displacement of a string, y(x), oscillating at frequency $\omega/(2\pi)$ satisfies

$$\mu^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \omega^2 y = 0, \tag{10}$$

where $\mu^2 = T/m$ and T is the tension of the string and m the mass per unit length. The string is fixed at x = 0 and x = L and thus the boundary conditions are y(0) = y(L) = 0. The general solution is

$$y = A\cos\frac{\omega x}{\mu} + B\sin\frac{\omega x}{\mu}.$$
(11)

Boundary condition y(0) = 0 implies A = 0 - but boundary condition y(L) = 0 can only be satisfied by a non-zero B if $\omega L = n\pi\mu$, where n is an integer. There are only solutions for given values of ω , $\omega_n = n\pi\mu/L$, and these are the *harmonics* of the string vibration.

2.1.6 Linear independence and the Wronskian

Two function are *linearly dependent* in an interval I if

$$\alpha f(t) + \beta g(t) = 0, \tag{12}$$

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for all $t \in I$ with α and β constants and not both zero. They are *linearly independent* if they are not linearly dependent. From (12) we also have

$$\alpha f'(t) + \beta g'(t) = 0. \tag{13}$$

Then solving between (12) and (13) for α and β we find that

$$\alpha \left(f(t)g'(t) - f'(t)g(t) \right) = 0 \quad \text{and} \quad \beta \left(f(t)g'(t) - f'(t)g(t) \right) = 0, \tag{14}$$

If not both of α and β vanish, then the Wronskian, W,

$$W(f,g) \equiv fg' - f'g = 0.$$
 (15)

If f(t) and g(t) are linearly dependent then W(f,g) = 0 for all $t \in I$. But it $W(f,g) \neq 0$ at some point $t = t_0$ ($t_0 \in I$), then f(t) and g(t) are linearly independent.

Abel's Theorem: Let $y_1(x)$ and $y_2(x)$ be two solutions of the second order differential equation L[y] = y'' + p(x)y' + q(x)y = 0, where p and q are continuous in the interval I. Then the Wronskian,

$$W(y_1, y_2) = c \exp\left(-\int^x p(s) \,\mathrm{d}s\right),\tag{16}$$

where c is a constant. Thus either W = 0 for all $x \in I$ or $W \neq 0$ for all $x \in I$. *Proof:* $y_2 L[y_1] - y_1 L[y_2] = y_2 y_1'' - y_1 y_2'' + p(y_1' y_2 - y_2' y_1)$. But $W' = y_1 y_2'' - y_1'' y_2$ and so

$$W' = -pW$$
 and then $W = c \exp\left(-\int^x p(s) \,\mathrm{d}s\right)$. (17)

The consequence is that $y_1(x)$ and $y_2(x)$ are linearly independent functions if the Wronskian does not vanish at some $x \in I$.

2.2 Inhomogeneous differential equations with constant coefficients

2.2.1 Form of the general solution

We consider a general inhomogeneous, second order, linear differential equation

$$L[y] \equiv \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\partial x} + q(x)y = f(x), \tag{18}$$

where in what follows the coefficients p(x) and q(x) are not necessarily constants. The homogeneous problem, L[y] = 0 has general solution $y(x) = \alpha y_1(x) + \beta y_2(x)$ for constants α and β . Suppose that $Y_1(x)$ and $Y_2(x)$ are both solutions of the inhomogeneous problem (18) then $L[Y_1(x)] = f(x)$ and $L[Y_2(x)] = f(x)$ and so $L[Y_1 - Y_2] = 0$. Thus $Y_1(x)$ and $Y_2(x)$ can only differ by a solution to the homogeneous problem. This means that the general solution to (18) may be expressed as

$$y = \alpha y_1(x) + \beta y_2(x) + y_p(x),$$
 (19)

where $y_p(x)$ is any particular solution to the inhomogeneous solution. Finding the general solution then amounts to finding a particular solution, $y_p(x)$ (also known as the *particular integral*) and finding solutions to the homogeneous problem (the latter are termed the *complementary function*).

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2.2.2 Finding a particular solution

Often guess work is easiest:

Example: $y'' + y = e^x$.

Try $y(x) = Ae^x$ and on substitution into the differential equation we find that $2Ae^x = e^x$ and so the particular integral is $y_p(x) = \frac{1}{2}e^x$.

Example: $y'' - 3y' + 2y = \sin x$.

Try $y = A \sin x + B \cos x$, $y' = A \cos x - B \sin x$, $y'' = -A \sin x - B \cos x$. On substitution into the differential equation, we find $(A + 3B) \sin x + (B - 3A) \cos x = \sin x$ and so $A = \frac{1}{10}$ and $B = \frac{3}{10}$.

This method of 'guessing' the particular integral works well if the function f(x) is (i) exponential; (ii) cos or sin; (iii) polynomial; and (iv) sums and products of (i)-(iii). If f(x) is not of this form, then use 'variation of parameters' (see §2.2.4).

2.2.3 Finding the general solution

The steps in forming the general solution to an inhomogeneous linear differential equation are:

- 1. Solve the homogeneous problem (L[y] = 0) to find the complementary function (e.g. $y(x) = \alpha y_1(x) + \beta y_2(x)$).
- 2. Find a particular solution to $L[y] = f(x), y = y_p(x)$.
- 3. Form the general solution $y(x) = \alpha y_1(x) + \beta y_2(x) + y_p(x)$.
- 4. Apply boundary conditions (if needed).

2.2.4 Variation of parameters

We want to find a particular solution to the linear differential equation

$$L[y] \equiv a(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = f(x).$$
(20)

Note that this method works for non-constant coefficients a(x), b(x) and c(x).

We suppose that we can find two linearly independent solutions to the homogeneous problem L[y] = 0, denoted by $y_1(x)$ and $y_2(x)$. We then seek a solution of the inhomogeneous problem (20) of the form $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$.

From this assumed form of solution, we find

$$y'_{p}(x) = v'_{1}y_{1} + v_{1}y'_{1} + v'_{2}y_{2} + v_{2}y'_{2} = v_{1}y'_{1} + v_{2}y'_{2},$$
(21)

if we choose $v'_1y_1 + v'_2y_2 = 0$. Then

$$y_p''(x) = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''.$$
(22)

On substitution into the governing equation, we find

$$L[y_p] = v'_1 y'_1 + v'_2 y'_2 = f/a.$$
(23)

Thus we deduce that

$$v_1' = \frac{-fy_2/a}{W(y_1, y_2)}$$
 and $v_2' = \frac{fy_1/a}{W(y_1, y_2)}$, (24)

where $W(y_1, y_2) \equiv y_1 y'_2 - y_2 y'_1$ is the Wronskian. This gives two first order differential equations for the functions $v_1(x)$ and $v_2(x)$ and from them the particular solution may be found.

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Non-trivial example: $y'' + y = \frac{1}{\sin x}$. Two linearly independent solutions of the homogeneous problem are $y_1 = \cos x$ and $y_2 = \sin x$. The Wronskian $W = \cos^2 x + \sin^2 x = 1$.

Then $v'_1 = -1$, which gives $v_1 = -x$. The integration constant can be set equal to zero, because if it were included then it would merely add a multiple of the complementary function to the particular integral.

Also $v'_2 = \cot x$ and so $v = \ln |\sin x|$. Hence the general solution is

$$y = -x\cos x + \ln|\sin x|\sin x + A\cos x + B\sin x.$$

2.2.5Application to mechanical systems

Suppose a mass m is suspended at the end of a spring of natural length L & spring constant k and is subject to a force F(t). The ensuing motion also experiences a drag force proportional to the velocity, u(t) and given by $\gamma u(t)$. The mass is at instantaneous position z(t). Applying Newton's second law, we find that



In equilibrium (no motion and forcing), we have $z = z_e$, given by $k(z_e - L) = mg$. We then write $z(t) = z_e + Z(t)$ and so the governing equation for Z(t) is

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}t^2} + \frac{\gamma}{m} \frac{\mathrm{d}Z}{\mathrm{d}t} + \frac{k}{m} Z = \frac{F(t)}{m}.$$
(26)

This is a second order linear differential equations with constant coefficients.

We first consider *free vibrations* when there is no forcing F(t) = 0. We look for a solution of the form $Z(t) = e^{rt}$ and so the characteristic equation is

$$r^{2} + \frac{\gamma}{m}r + \frac{k}{m} = \left(r + \frac{\gamma}{2m}\right)^{2} - \frac{\gamma^{2} - 4mk}{4m^{2}} = 0.$$
 (27)

There are complex roots to (27) if $\gamma^2 < 4mk$ and real roots if $\gamma^2 > 4mk$. This determines whether the solution is oscillatory or not.

We now consider forced vibrations, with $F(t) = F_0 \cos \Omega t$. For simplicity, we treat the undamped response ($\gamma = 0$), starting from rest (Z(0) = 0, dZ/dt(0) = 0). The complementary function of (26) is $Z = A \cos \omega t + B \sin \omega t$, where $\omega^2 = k/m$. The particular integral is

$$Z(t) = \frac{F_0}{m\left(\omega^2 - \Omega^2\right)} \cos \Omega t$$

Thus the solution satisfying the initial conditions is

$$Z(t) = \frac{F_0}{m\left(\omega^2 - \Omega^2\right)} \left(\cos\Omega t - \cos\omega t\right).$$
(28)

and using double-angle formulae, this may be written as

$$Z(t) = \frac{-2F_0}{m\left(\omega^2 - \Omega^2\right)} \sin\left(\frac{\left(\Omega + \omega\right)t}{2}\right) \sin\left(\frac{\left(\Omega - \omega\right)t}{2}\right).$$
(29)

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Figure 1: The response, Z(t), to forcing $F(t) = F_0 \cos \Omega t$ with $F_0 = 1$, m = 1, $\omega = 1$ and $\Omega = 1.1$. The dotted lines depict the variation of the amplitude.

This situation may be interpreted as a signal with frequency $(\Omega + \omega)/2$ and amplitude of $2F_0/[m(\omega^2 - \Omega^2)]$ which varies $\sin((\Omega - \omega)t/2)$. When the forcing frequency Ω is close to the natural frequency, ω , the amplitude is large and the response is known as the phenomenon of 'beats'.

What happens if the forcing frequency is equal to the natural frequency $(\Omega = \omega)$? In this case the solution derived above is invalid, because the particular integral is already accounted for in the complementary function. Instead we seek a particular solution of the form $Z_p = Ct \sin \omega t$ and on substitution we find $C = F_0/[2m\omega]$. Applying the initial conditions, we deduce that

$$Z(t) = \frac{F_0 t}{2m\omega} \sin \omega t.$$
(30)

Thus the amplitude of the response grows as $F_0 t/[2m\omega]$. This phenomenon is known as resonance.



Figure 2: The resonant response, Z(t), to forcing $F(t) = F_0 \cos \omega t$ with $F_0 = 1$, m = 1 and $\omega = 1$. The dotted lines depict the growth of the amplitude.