3 Discrete Dynamical Systems

3.1 Definitions

A first-order discrete dynamical system is a map by which $u(n+1)$ is determined as a function of $u(n)$,

$$u(n+1) = f(u(n)), \tag{1}$$

where n is a positive integer. Given $u(0)$, this map generates a unique sequence $u(n)$. These maps are also known as *difference equations.*
A first order *affine* map is of the form

$$u(n+1) = \alpha u(n) + \beta, \tag{2}$$

where α and β are constants.

3.2 General theory of first-order affine difference equations

Equilibrium value:
Suppose $u(n) = E$ then $E = \alpha E + \beta$ and so $E = \beta/(1-\alpha)$ (provided $\alpha \neq 1$).

If $\alpha = 1$ then $u(n+1) = u(n) + \beta = u(n-1) + 2\beta = u(0) + n\beta$, but for equilibrium $u(n+1) = u(n)$ and so $\beta = 0$. Thus the map is $u(n+1) = u(n)$ and so all values are equilibrium values.
General solution:
The solution is composed of a sum of the solution to the homogeneous problem $(u(n+1) = \alpha u(n))$, plus a particular solution.
The homogeneous problem $u(n+1) = \alpha u(n)$ has solution $u(n) = A\alpha^n$, where A is a constant, while $u(n) = E$ is a particular solution. Thus the general solution is

$$u(n) = A\alpha^n + \frac{\beta}{1-\alpha}. \tag{3}$$

The constant, A, is determined from the initial value $u(0)$.
Stability:
If $|\alpha| < 1$ then $|\alpha^n| \to 0$ as $n \to \infty$ and so $u(n) \to E$. This situation is *stable* because the equilibrium value is approached asymptotically.
Conversely if $|\alpha| > 1$ then the equilibrium value is *unstable.*

3.2.1 More general consideration of stability

Suppose the map is not affine and so more generally $u(n+1) = f(u(n))$. The equilibrium values satisfies $E = f(E)$. Close to the equilibrium value we denote $u(n) = E + \delta_n$ where δ_n is small. Then

$$u(n+1) \equiv E + \delta_{n+1} = f(E + \delta_n) = E + \delta_n \left.\frac{\mathrm{d}f}{\mathrm{d}u}\right|_{u=E} + \dots \tag{4}$$

Hence $\delta_{n+1} = \delta_n \dfrac{\mathrm{d}f}{\mathrm{d}u}(E)$. Then $|\delta_{n+1}| < |\delta_n|$ if $\left|\dfrac{\mathrm{d}f}{\mathrm{d}u}(E)\right| < 1$ and in this case the equilibrium point is *stable.* Conversely $|\delta_{n+1}| > |\delta_n|$ if $\left|\dfrac{\mathrm{d}f}{\mathrm{d}u}(E)\right| > 1$ and then the equilibrium point is *unstable.*

3.3 Web diagrams

These are a convenient graphical way of representing the evolution of a map.
Consider the example

$$u(n+1) = \frac{1}{2}u(n) + 1. \tag{5}$$

We can read this as

$$u(next) = \frac{1}{2}u(now) + 1. \tag{6}$$

In order to understand the evolution we draw the lines

$$y = x \qquad \text{and} \qquad y = \frac{1}{2}x + 1. \tag{7}$$

To see how the map moves we use the rule:

1. First move vertically from $y = x$, the point $(u(now), u(now))$, to $y = \frac{1}{2}x + 1$, the point $(u(now), u(next))$
2. Then move horizontally from $y = \frac{1}{2}x + 1$, the point $(u(now), u(next))$, to $y = x$, the point $(u(next), u(next))$.
3. Repeat.

This process is illustrated in figure 1.

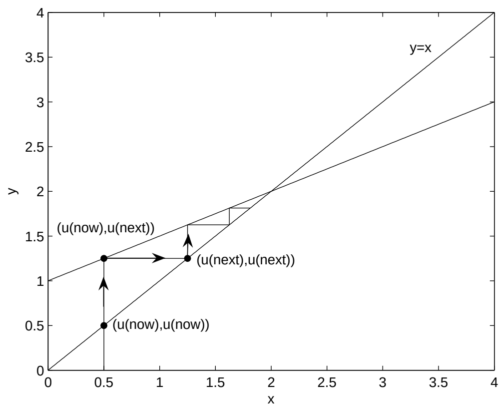

Figure 1: Web diagram for map $u(n+1) = \frac{1}{2}u(n) + 1$ with $u(0) = \frac{1}{2}$

We see that the intersection of the two lines $(2, 2)$ corresponds to the equilibrium point $u(n) = 2$. Also in this case whether we start with $u(0)$ larger than, or smaller than, the equilibrium point, we move towards it. This is because for this map $|\alpha| = \frac{1}{2} < 1$, thus the equilibrium point is stable.

Figure 2 illustrates the example

$$u(n+1) = 2u(n) - 2. \tag{8}$$

In this case we draw

$$y = x \qquad \text{and} \qquad y = 2x - 2. \tag{9}$$

We see that we evolve away from the point $(2, 2)$ corresponding to the equilibrium value $u(n) = 2$. This equilibrium point is unstable.

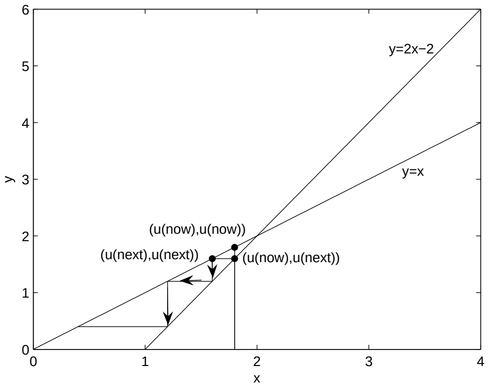

Figure 2: Web diagram for map $u(n+1) = 2u(n) - 2$ with $u(0) = \frac{9}{5}$

In summary: with a map $u(n+1) = \alpha u(n) + \beta$ we can associate a line $y = \alpha x + \beta$. The equilibrium point is where $y = \alpha x + \beta$ intersects $y = x$. The equilibrium point is stable if, at the intersection, the line $y = \alpha x + \beta$ crosses $y = x$ in the sector between -45^o and $+45^o$. If it does not, the point is unstable.

If $y = \alpha x + \beta$ at exactly -45^o, the evolution is periodic with period 2, since in this case the map is $u(n+1) = -u(n) + \beta$ and we can see $u(n+2) = u(n)$.

3.4 Solutions to more general linear difference equations

3.4.1 Higher order equations

A second order, homogeneous, linear difference equation is given by

$$au(n+2) + bu(n+1) + cu(n) = 0, \tag{10}$$

where a, b and c are constants. We seek a solution by writing $u(n) = A\lambda^n$, then

$$a\lambda^2 + b\lambda + c = 0. \tag{11}$$

This quadratic equation has two distinct roots, provided $b^2 \neq 4ac$, which we denotes λ_1 and λ_2. the general solution is then given by

$$u(n) = A\lambda_1^n + B\lambda_2^n, \tag{12}$$

where A and B are constants. If there is a repeated root then

$$u(n) = A\lambda^n + Bn\lambda^n. \tag{13}$$

This methodology carries over to higher order systems.

3.4.2 Inhomogeneous difference equations

For example

$$u(n+1) - \alpha u(n) = f(n). \tag{14}$$

The methodology to find the general solution is first to solve the homogeneous problem ($f = 0$) and then to find a particular solution. For simple functions, $f(n)$, the easiest strategy for establishing a particular solution is to guess a form of solution and then 'tune' the guess until it can be made to satisfy the equation.

3.5 Nonlinear difference equations: The logistic map

The logistic map takes the form

$$u(n+1) = au(n)(1 - u(n)), \tag{15}$$

where a is a constant.

We extend the idea of the previous sections to general maps: with the map $u(n+1) = f(u(n))$ we can associate a curve $y = f(x)$. The equilibrium points are where $y = f(x)$ intersects $y = x$. We can understand whether an equilibrium point is stable or unstable by considering whether at the intersection, the curve $y = f(x)$ crosses $y = x$ in the sector between -45^o and $+45^o$ or not. If the former then the equilibrium point is stable, otherwise unstable. We should use a little care here: when we say a point is stable, we mean that if we start *near to* the point then we will get closer as n increases [similarly a point is unstable if we move away from it, having started near to it].

In considering the map (15), we first note that with $0 \le a \le 4$, if $0 \le u(0) \le 1$, then $0 \le u(n) \le 1$ for all n. We will be interested in evolutions that start with $0 \le u(0) \le 1$.

An equilibrium, or *fixed*, point satisfies

$$x = ax(1 - x) \qquad \Rightarrow \qquad x = 0 \quad \text{or} \quad x = 1 - \frac{1}{a}. \tag{16}$$

At the equilibrium point $f'(0) = a$ and $f'(1 - 1/a) = 2 - a$.

3.5.1 $0 < a < 1$

In this case the only fixed point in the region $0 \le x \le 1$ is $x = 0$, see Figure 3. It is not difficult to see [for example using a web diagram or analytically] that $x = 0$ is a stable fixed point.

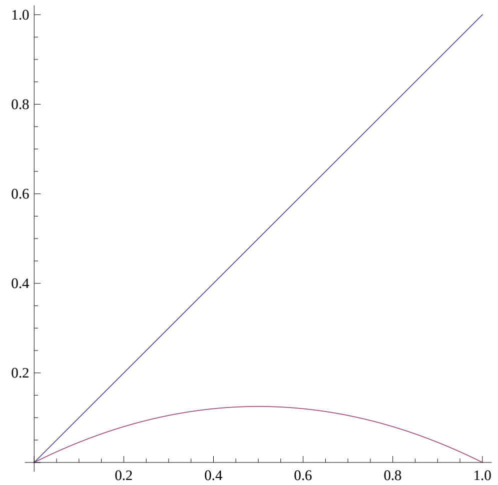

Figure 3: $y = \frac{1}{2}x(1 - x)$ showing that the only point of intersection with $y = x$ in the region $0 \le x \le 1$ is $x = 0$

3.5.2 $1 < a < 3$

As we increase a to become greater than 1, we see a second fixed point in the region $0 \le x \le 1$, since now the fixed point $1 - \frac{1}{a}$ is between 0 and 1. For example consider $a = 5/2$, see Figure 4. Then the fixed points are $x = 0$ and $x = 3/5$. To find out whether the fixed point $x = 3/5$ is stable or unstable we look at the derivative of $y = \frac{5}{2}x(1 - x)$ at $x = 3/5$ [the point of intersection

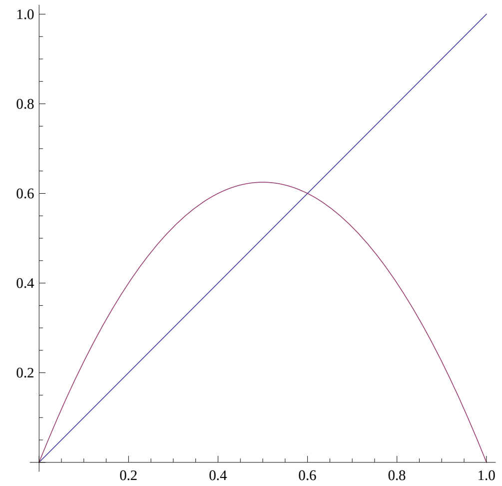

Figure 4: $y = \frac{5}{2}x(1-x)$ showing the two points of intersection with $y = x$

of $y = \frac{5}{2}x(1-x)$ with $y = x$.] This is given by $f'(3/5) = -1/2$ - and so the equilibrium point $x = 3/5$ is stable.

We can also see this by considering a point close to the fixed point and seeing where it evolves to under the map. Let $u(0) = 3/5 + \delta$ then

$$u(1) = \frac{5}{2}u(0)(1-u(0)) = \frac{5}{2}(\frac{3}{5}+\delta)(\frac{2}{5}-\delta) \simeq \frac{3}{5} - \frac{1}{2}\delta. \tag{17}$$

Thus for small δ, $u(1)$ is closer to the fixed point than $u(0)$ was: the fixed point is stable.

3.5.3 $a > 3$

As we increase a beyond 3 we see that the modulus of the gradient at the fixed point, $2-a$, becomes greater than 1: the fixed point has become unstable.

For example consider $a = 10/3$. The gradient at the fixed point $x = 7/10$ of $y = \frac{10}{3}x(1-x)$ is $-4/3$.

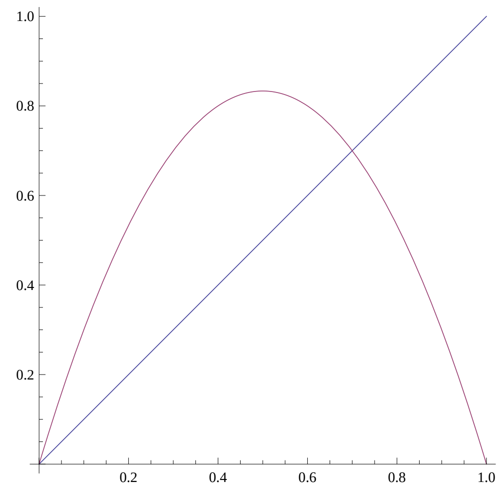

Figure 5: $y = \frac{10}{3}x(1-x)$ showing that the point $x = 7/10$ is an unstable fixed point

We note that $u(n)$ stays between 0 and 1 for all n [if our $0 \leq u(0) \leq 1$], so $u(n)$ cannot become infinitely large as it did for affine maps with an unstable fixed point.

In order to see what happens we consider two steps of evolution. Thus we consider $u(n+2)$ as a function of $u(n)$.

$$u(n+2) = F_a^{(2)}(u(n)) = a^2 u(n)\left(1-u(n)\right)\left(1-au(n)\left(1-u(n)\right)\right). \tag{18}$$

This is a quartic polynomial in $u(n)$.

Consider first the case $a = 5/2$, see Figure 6. In this case we know that there is a fixed point $x = 3/5$; this must also be a fixed point of the map $F_{5/2}^{(2)}$. Indeed $x = 0$ and $x = 3/5$ are the only real points of intersection of $y = x$ with $y = a^2x(1-x)(1-ax(1-x))$ when $a = 5/2$:

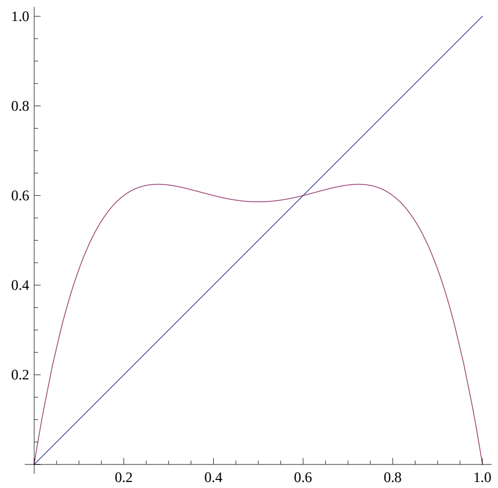

Figure 6: $y = F_{5/2}^{(2)}(x)$ showing two points of intersection with $y = x$ between 0 and 1

However as a increases above 3, as well as the two fixed points 0 and $1 - 1/a$ we get two new fixed points, see Figure 7. i.e. there are two new points of intersection of $y = x$ with $y = a^2x(1-x)(1-ax(1-x))$ since the equation for fixed points of $F_a^{(2)}$ is the quartic:

$$x = a^2x(1-x)(1-ax(1-x)). \tag{19}$$

The two new solutions are

$$x = \frac{1+a+\sqrt{(a-3)(a+1)}}{2a} \quad \text{and} \quad \frac{1+a-\sqrt{(a-3)(a+1)}}{2a}. \tag{20}$$

In the case $a = 10/3$, for example, the four fixed points are thus

$$x = 0,\ \frac{7}{10},\ \frac{13+\sqrt{13}}{20},\ \frac{13-\sqrt{13}}{20}. \tag{21}$$

We note that for $a = 10/3$, if $u(n) = \frac{13+\sqrt{13}}{20}$ then $u(n+1) = \frac{13-\sqrt{13}}{20}$ and $u(n+2) = \frac{13+\sqrt{13}}{20}$. Thus the points $x = \frac{13\pm\sqrt{13}}{20}$ are *fixed points of order two*. Indeed (20) are fixed points of order 2 for general values of a between 3 and 4.

For $a = 10/3$, the fixed point at $x = 7/10$ is unstable since consider $u(0) = 7/10 + \delta$; then

$$u(1) = \frac{10}{3}u(0)(1-u(0)) = \frac{10}{3}(\frac{7}{10}+\delta)(\frac{3}{10}-\delta) =\simeq \frac{7}{10} - \frac{4}{3}\delta. \tag{22}$$

So what has happened as we increase a is that the initially stable fixed point becomes unstable at the same time creating two new fixed points, these latter points being fixed points of order two.

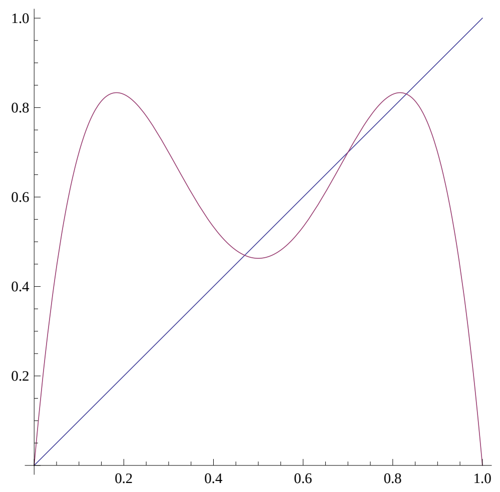

Figure 7: $y = F^{(2)}_{10/3}(x)$ showing four points of intersection with $y = x$ between 0 and 1

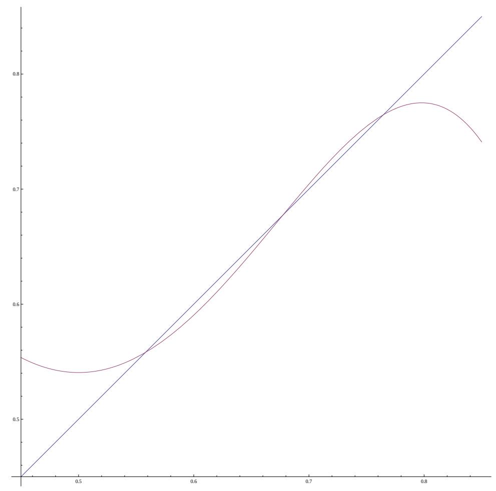

Figure 8: $y = F^{(2)}_{3.1}(x)$. As a has increased to just above 3, the new fixed points that have appeared are stable

This phenomenon of multiple fixed points emerging from a single one as a parameter increases is called *bifurcation*. From Figure 8 we can see that initially [i.e. for a just above 3], these new fixed points of $F^{(2)}$ are stable since the graph of $y = F^{(2)}(x)$ crosses the line $y = x$ in the stable section between $\pm 45^o$.

We can also study the stability of the fixed points of order two analytically in a similar way to the way we looked at fixed points of order 1. In this case

$$\frac{\mathrm{d}F_a^{(2)}}{\mathrm{d}x} = 4 + 2a - a^2, \tag{23}$$

at the equilibrium points of order two. Thus these points are stable for $3 < a < 1 + \sqrt{6}$.

As a increases further these two fixed points of order two themselves become unstable creating four new fixed points of order 4, leading to a hierarchy of bifurcations . The review article by R.M. May (*Simple Mathematical Models with very complicated dynamics*, Nature 261, 1976) gives further interesting reading about this topic. As a increases one eventually reaches a point, $a \simeq 3.5700$, beyond which there are an infinite number of fixed points; this has been christened the "chaotic" region.

3.5.4 Sensitivity to initial conditions

A further manifestation of chaos is that the eventual evolution is sensitive to initial conditions; this is studied numerically in Sheet 12 question 5. It is rather difficult to see this analytically for general a in the chaotic region. However $a = 4$ is one of the few values for which an exact analytic solution is known, and we can see this behaviour in the values of $u(n)$ for this value of a.

One can check by direct substitution that the solution to $u(n+1) = 4u(n)(1-u(n))$ is

$$u(n) = \sin^2(2^n\theta\pi) \quad \text{where} \quad \theta = \frac{1}{\pi}\sin^{-1}\sqrt{u(0)}. \tag{24}$$

So consider two different initial conditions with values of θ of θ_0 and $\theta_0 + \epsilon$. In the first case we get

$$u(n) = \sin^2(2^n\theta_0\pi). \tag{25}$$

In the second case we get

$$u(n) = \sin^2(2^n\theta_0\pi + 2^n\pi\epsilon). \tag{26}$$

So we see that however small ϵ is, after about $\log_2(1/\epsilon)$ time-steps, $2^n\epsilon\pi$ is of order π so the two evolutions will be quite different. We note that the difference between the two angles $2^n\theta_0\pi$ and $2^n\theta_0\pi + 2^n\epsilon\pi$ increases exponentially with n.