Continuous dynamical systems: 4 coupled first order differential equations

We focus on systems with two dependent variables so that

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f(x_1, x_2, t) \qquad \text{and} \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = g(x_1, x_2, t).$$

Most of the analysis will be for *autonomous systems* so that

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f(x_1, x_2) \qquad \text{and} \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = g(x_1, x_2). \tag{1}$$

A useful compact notation is to write $\mathbf{x} = (x_1(t), x_2(t))$ and $\mathbf{f} = (f, g)$ so that $\frac{d\mathbf{x}}{dt} = \mathbf{f}$.

Equilibrium points 4.1

These are *fixed points* of the system where

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = 0. \tag{2}$$

They are found by simultaneously solving $f(x_1, x_2) = 0$ and $g(x_1, x_2) = 0$. For nonlinear functions, f and q, there may be more than one fixed point.

4.2Linear systems

In general a linear system with constant coefficients can be written as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x},\tag{3}$$

where M is a matrix of constant coefficients.

Superposition of solutions 4.2.1

If $\mathbf{x_1}$ and $\mathbf{x_2}$ are both solutions to the linear system (3), then

$$\mathbf{x} = \alpha \mathbf{x_1} + \beta \mathbf{x_2},$$

is also a solution. Proof: $\frac{d\mathbf{x}}{dt} = \alpha \frac{d\mathbf{x_1}}{dt} + \beta \frac{d\mathbf{x_2}}{dt} = \alpha M \mathbf{x_1} + \beta M \mathbf{x_2} = M \mathbf{x}.$

This means that to find the general solution of (3), we need to find sufficient linearly independent solutions and then form arbitrary additive combinations of them. For a second-order system, we need two linearly independent solutions.

4.2.2Constructing the solution

We seek a solution of the form $\mathbf{x}(t) = \mathbf{a}e^{\lambda t}$. This is a solution to (3) if

$$\lambda \mathbf{a} \mathrm{e}^{\lambda t} = M \mathbf{a} \mathrm{e}^{\lambda t}.\tag{4}$$

But since $e^{\lambda t} > 0$, for a non-trivial solution, we require

$$\det\left(M - \lambda I\right) = 0. \tag{5}$$

©University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.

Then for each solution λ_i , we can find a vector $\mathbf{a}^{(i)}$ such that

$$(M - \lambda_i I)\mathbf{a}^{(i)} = 0. \tag{6}$$

This is an eigenvalue problem: λ_i are the eigenvalues and $\mathbf{a}^{(i)}$ are the eigenvectors. The solution for $\lambda = \lambda_i$ is then

$$\mathbf{x}(t) = \mathbf{a}^{(i)} \mathrm{e}^{\lambda_i t},\tag{7}$$

These are sometimes termed the *fundamental solutions*.

Example 1: $\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We seek a solution $(x_1, x_2) = (A, B)e^{\lambda t}$ and so the eigenvalue problem is

$$\left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$
(8)

A non-trivial solution $(A \neq 0 \text{ and } B \neq 0)$ requires

$$\begin{vmatrix} 1-\lambda & 1\\ 4 & 1-\lambda \end{vmatrix} = 0 \quad \text{which implies} \quad (1-\lambda)^2 - 4 = 0.$$
(9)

Thus
$$\lambda = 1 \pm 2$$
. So there are two real and distinct values.
When $\lambda = 3$, $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
When $\lambda = -1$, $\begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = \beta \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
So the general solution is
 $\mathbf{x}(t) = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + \beta \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$ (10)

The constants α and β can be determined from initial conditions.

The **phase plane** is a useful geometric way of viewing the solutions from many initial conditions. The phase plane is the plane (x(t), y(t)) and each curve in it denotes a solution associated with a particular initial condition. The curves are termed 'trajectories' and each has an arrow to show the direction of evolution as t increases.



Figure 1: The phase plane for example 1.

The phase plane for example 1 is plotted in figure 1. We note that the origin is a fixed point (an equilibrium point) of this system, but there are no trajectories that remain close to the origin as $t \to \infty$. Thus this fixed point is *unstable*.

©University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.

4.2.3 Possible forms of solution: $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

For a 2-D system the eigenvalue equation $|M - \lambda I| = 0$ is a quadratic equation. For a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this equation is given by

$$\lambda^2 - (a+d)\lambda + ad - bc = 0. \tag{11}$$

In general this quadratic equation will have two distinct roots, λ_1 and λ_2 , unless $4(ad - bc) = (a + d)^2$. If the solutions are complex valued, then λ_1 and λ_2 are complex conjugates because M is real-valued. The general solution is then

$$\mathbf{x}(t) = c_1 \mathbf{a}_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathbf{a}_2 \mathrm{e}^{\lambda_2 t}.$$
 (12)

It $Re{\lambda_1} < 0$ and $Re{\lambda_2} < 0$, then $|\mathbf{x}| \to 0$ as $t \to \infty$. This implies that the fixed point $\mathbf{x} = (0,0)$ is *stable*. It $Re{\lambda_1} > 0$ or $Re{\lambda_2} > 0$, then $|\mathbf{x}| \to \infty$ as $t \to \infty$. This implies that the fixed point $\mathbf{x} = (0,0)$ is unstable.

There is a classification of the fixed points depending on the values of λ_1 and λ_2 :

- 1. λ_1 and λ_2 real valued:
 - $\lambda_1 < 0$ and $\lambda_2 < 0$: stable node.
 - $\lambda_1 > 0$ and $\lambda_2 > 0$: unstable node.
 - $\lambda_1 \lambda_2 < 0$: saddle.

2. λ_1 and λ_2 complex conjugates, so that $\lambda_1 = \mu + i\omega$ and $\lambda_2 = \mu - i\omega$:

- $\mu > 0$: unstable spiral.
- $\mu < 0$: stable spiral.
- $\mu = 0$: centre.

Example 2: $\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We seek a solution $(x_1, x_2) = (A, B) \mathrm{e}^{\lambda t}$ and so the eigenvalue problem is

$$\begin{bmatrix} \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$
(13)

A non-trivial solution $(A \neq 0 \text{ and } B \neq 0)$ requires

$$\begin{vmatrix} -3 - \lambda & \sqrt{2} \\ \sqrt{2} & -2 - \lambda \end{vmatrix} = 0 \quad \text{which implies} \quad (\lambda + 4)(\lambda + 1) = 0.$$
 (14)

Thus $\lambda = -4$ and $\lambda = -1$. So the origin (the fixed point) is a stable node.

When
$$\lambda = -4$$
, $\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = \alpha \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$.
When $\lambda = -1$, $\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = \beta \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

[©]University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.



Figure 2: The phase plane for example 2.

So the general solution is

$$\mathbf{x}(t) = \alpha \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} + \beta \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}.$$
 (15)

The phase plane for example 2 is plotted in figure 2. We note that the origin is a stable fixed point of this system, because all the trajectories approach it as $t \to \infty$.

Example 3: $\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We seek a solution $(x_1, x_2) = (A, B)e^{\lambda t}$ and so the eigenvalue problem is

$$\left[\begin{pmatrix} 0 & \alpha^2 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$
 (16)

A non-trivial solution $(A \neq 0 \text{ and } B \neq 0)$ requires

$$\begin{vmatrix} -\lambda & \alpha^2 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \text{which implies} \quad \lambda^2 + \alpha^2 = 0.$$
 (17)

Thus $\lambda = \pm i\alpha$ - so the origin (the fixed point) is a centre and we will show that the trajectories about it are closed curves.

When
$$\lambda = i\alpha$$
, $\begin{pmatrix} -i\alpha & \alpha^2 \\ -1 & -i\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = c_1 \begin{pmatrix} -i\alpha \\ 1 \end{pmatrix}$.
When $\lambda = -i\alpha$, $\begin{pmatrix} i\alpha & \alpha^2 \\ -1 & i\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = c_2 \begin{pmatrix} i\alpha \\ 1 \end{pmatrix}$.
So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -\mathrm{i}\alpha\\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\alpha t} + c_2 \begin{pmatrix} \mathrm{i}\alpha\\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha t}.$$
 (18)

By introducing $c_3 = c_1 + c_2$ and $c_4 = i(c_2 - c_1)$, this can be re-written

$$\mathbf{x}(t) = c_3 \begin{pmatrix} \alpha \sin \alpha t \\ \cos \alpha t \end{pmatrix} + c_4 \begin{pmatrix} \alpha \cos \alpha t \\ -\sin \alpha t \end{pmatrix}.$$
 (19)

The trajectories in the phase plane are easy to deduce analytically and are given by

$$x_1^2 + \alpha^2 x_2^2 = \text{constant.}$$
 (20)

[©]University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.



Figure 3: The phase plane for example 3 with $\alpha = 2$

They are ellipses and are plotted in figure 3. **Example 4**: $\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We seek a solution $(x_1, x_2) = (A, B)e^{\lambda t}$ and a non-trivial solution $(A \neq 0 \text{ and } B \neq 0)$ requires

$$\begin{vmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{vmatrix} = 0 \quad \text{which implies} \quad (\lambda + \frac{1}{2})^2 + 1 = 0.$$
(21)

Thus $\lambda = -\frac{1}{2} \pm i$ - so the origin (the fixed point) is a stable spiral. When $\lambda = -\frac{1}{2} + i$, $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = c_1 \begin{pmatrix} -i \\ 1 \end{pmatrix}$. When $\lambda = -\frac{1}{2} - i$, $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ and so $\begin{pmatrix} A \\ B \end{pmatrix} = c_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$. So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix} e^{(-\frac{1}{2} + \mathbf{i})t} + c_2 \begin{pmatrix} \mathbf{i} \\ 1 \end{pmatrix} e^{(-\frac{1}{2} - \mathbf{i})t}.$$
 (22)

By re-defining the constants, this can be re-written

$$\mathbf{x}(t) = e^{-t/2} \left[c_3 \left(\frac{\sin t}{\cos t} \right) + c_4 \left(-\frac{\cos t}{\sin t} \right) \right].$$
(23)

The phase plane for example 4 is plotted in figure 4. We note that the origin is a stable fixed point of this system, because all the trajectories approach it as $t \to \infty$.

4.3 Nonlinear coupled first-order systems

For the non-linear system $\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{pmatrix}$, we can find *fixed* points by simultaneously solving f = 0 and g = 0. But how do we determine the nature and stability of the fixed points? The important idea is the examine the behaviour sufficiently close to a fixed point and treat the system as linear system in this region. The stages in analysing the system are: (i) find the fixed points; (ii) linearise the equations close to the fixed points; (iii) classify the linearised system.



Figure 4: The phaseplane for example 4.

4.3.1 Example: Predator-Prey systems

x(t) denotes the population of the prey, while y(t) denotes the population of the predator. Their interaction can be modelled by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{2}xy,\tag{24}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{3}{4}y + \frac{1}{4}xy,\tag{25}$$

and we are interested only in the region x > 0 and y > 0.

First we find the equilibrium points, $x(1 - \frac{1}{2}y) = 0$ and $y(-\frac{3}{4} + \frac{1}{4}x) = 0$. These are at (x, y) = (0, 0) and (x, y) = (3, 2). We examine each in turn.

Close to (x, y) = (0, 0), we linearise the equations on the assumption that $|x| \ll 1$ and $|y| \ll 1$ so that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \dots$$
 and $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{3}{4}y + \dots$ (26)

This may be integrated immediately to give

$$x(t) = c_1 e^t$$
 and $y(t) = c_2 e^{-3t/4}$. (27)

Thus the eigenvalues of the linear system at $\lambda = 1$ and $\lambda = -\frac{3}{4}$. The point (0,0) is a saddle point and is therefore unstable.

Trajectories in the phase plane are given by $x^3y^4 = \text{constant}$, y sufficiently close to the origin and shown in the sketch.

Close to (x, y) = (3, 2), we write x(t) = 3 + X(t) and y(t) = 2 + Y(t) and examine the governing equations when $|X(t)| \ll 1$ and $|Y(t)| \ll 1$. This gives

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 3 + X - \frac{1}{2}(3+X)(2+Y) = -\frac{3}{2}Y + \dots$$
(28)

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{3}{4}(2+Y) + \frac{1}{4}(3+X)(2+Y) = \frac{1}{2}X + \dots$$
(29)

In matrix form, this linearised system may be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} X\\ Y \end{pmatrix} = \begin{pmatrix} 0 & -3/2\\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} X\\ Y \end{pmatrix}. \tag{30}$$



©University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.

The eigenvalues are therefore given by $\begin{vmatrix} -\lambda & -3/2 \\ 1/2 & -\lambda \end{vmatrix} = 0$, which implies $\lambda = \pm i\sqrt{3}/2$. The fixed point is therefore a *centre*.

When
$$\lambda = i\sqrt{3}/2$$
, $\begin{pmatrix} -i\sqrt{3}/2 & -3/2 \\ 1/2 & -i\sqrt{3}/2 \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ -i/\sqrt{3} \end{pmatrix}$.
When $\lambda = -i\sqrt{3}/2$, $\begin{pmatrix} i\sqrt{3}/2 & -3/2 \\ 1/2 & i\sqrt{3}/2 \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ i/\sqrt{3} \end{pmatrix}$.
So the general solution close to the fixed point is given by

$$\begin{pmatrix} X\\ Y \end{pmatrix} = c_1 \begin{pmatrix} 1\\ -i/\sqrt{3} \end{pmatrix} e^{i\sqrt{3}t/2} + c_2 \begin{pmatrix} 1\\ i/\sqrt{3} \end{pmatrix} e^{-i\sqrt{3}t/2}, \qquad (31)$$
$$\begin{pmatrix} \cos(\sqrt{3}t/2) \end{pmatrix} = \begin{pmatrix} \sin(\sqrt{3}t/2) \end{pmatrix}$$

$$= c_3 \begin{pmatrix} \cos(\sqrt{3t/2}) \\ \sin(\sqrt{3t/2})/\sqrt{3} \end{pmatrix} + c_4 \begin{pmatrix} \sin(\sqrt{3t/2}) \\ \cos(\sqrt{3t/2})/\sqrt{3} \end{pmatrix}.$$
 (32)

Then using double-angle formulae, these may be written as y

$$X = R\cos\left(\frac{\sqrt{3}t}{2} + \phi\right) \quad \text{and} \quad Y = \frac{R}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2} + \phi\right).$$
(33)

where R and ϕ are constants. This implies that close to the fixed points the populations of predators and prey oscillate $\pi/2$ out of phase with each other and that in the phase plane, the trajectories are ellipses $X^2 + 3Y^2 = \text{constant}$, as sketched.

Now that we have analysed the motion in the vicinities of the two fixed points, we can sketch the entire phase plane. This is shown in figure 5.



Figure 5: Trajectories in the phase plane for a predator-prey model

In this case the exact trajectories may be computed directly, because

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x} \frac{(-3+x)}{(2-y)},$$

which may be integrated to give

$$\log|y^4x^3| - 2y - x = \text{constant.}$$
(34)

©University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.



Example: Competing species 4.3.2

A model of the population of two species with populations, x(t) and y(t), which are competing for resources is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x-y),\tag{35}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(\frac{3}{4} - y - \frac{x}{2}\right). \tag{36}$$

The fixed points are determined by simultaneously solving x(1-x-y) = 0 and $y(\frac{3}{4}-y-\frac{x}{2}) = 0$, which yields 4 fixed points, $(x, y) = (0, 0), (0, \frac{3}{4}), (1, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. To characterise the phase plane, we need to systematically investigate the nature of the system in the vicinity of each of them.

Close to the fixed point at (0,0), the linearised system becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + \dots$$
 and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3}{4}y + \dots,$ (37)

which may be integrated to give $x(t) = c_1 e^t$ and $y(t) = c_2 e^{3t/4}$. Thus the origin is an unstable node.

Close to the fixed point at $(0, \frac{3}{4})$, we write $y = \frac{3}{4} + Y(t)$ and the linearised system becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x - \frac{3}{4} - Y) = \frac{1}{4}x + \dots$$
(38)

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{3}{4} + Y\right) \left(\frac{3}{4} - \frac{3}{4} - Y - \frac{x}{2}\right) = -\frac{3Y}{4} - \frac{3x}{8}.$$
(39)

Seeking a solution of the form $(x, Y) = \mathbf{a} e^{\lambda t}$, requires that

$$\begin{vmatrix} 1/4 - \lambda & 0 \\ -3/8 & -3/4 - \lambda \end{vmatrix} = 0, \quad \text{which implies} \quad \lambda = \frac{1}{4}, -\frac{3}{4}$$

This fixed point is therefore a saddle. The eigenvectors are given by: when $\lambda = 1/4$, $\begin{pmatrix} 0 & 0 \\ -3/8 & -1 \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ -\frac{3}{8} \end{pmatrix}$. when $\lambda = -3/4$, $\begin{pmatrix} -1/2 & 0 \\ -3/8 & 0 \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The general solution close to $(0, \frac{3}{4})$ is therefore

$$\begin{pmatrix} x \\ y - \frac{3}{4} \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ -\frac{3}{8} \end{pmatrix} e^{t/4} + c_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t/4}.$$
 (40)

Close to the fixed point at (1,0), we write x = 1 + X(t) and the linearised system becomes

$$\frac{\mathrm{d}X}{\mathrm{d}t} = (1+X)(-X-Y) = -X - y + \dots$$
(41)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(\frac{3}{4} - y - \frac{1}{2} - \frac{X}{2}\right) = \frac{y}{4}.$$
(42)

Seeking a solution of the form $(X, y) = \mathbf{a}e^{\lambda t}$, requires that

$$\begin{vmatrix} -1 - \lambda & -1 \\ 0 & 1/4 - \lambda \end{vmatrix} = 0$$
, which implies $\lambda = \frac{1}{4}, -1$

[©]University of Bristol 2013. This material is the copyright of the University unless explicitly stated otherwise. It is provided exclusively for educational purposes at the University and is to be downloaded or copied for your private study only.

This fixed point is therefore a saddle. The eigenvectors are given by:

when $\lambda = 1/4$, $\begin{pmatrix} -5/4 & -1 \\ 0 & 0 \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ -5/4 \end{pmatrix}$. when $\lambda = -1$, $\begin{pmatrix} 0 & -1 \\ 0 & 3/4 \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The general solution cloe to (1, 0) is therefore

$$\binom{x-1}{y} = c_5 \begin{pmatrix} 1\\ -\frac{5}{4} \end{pmatrix} e^{t/4} + c_6 \begin{pmatrix} 1\\ 0 \end{pmatrix} e^{-t}.$$
(43)

Close to the fixed point at $(\frac{1}{2}, \frac{1}{2})$, we write $x = \frac{1}{2} + X(t)$ and $y(t) = \frac{1}{2} + Y(t)$ and the linearised system becomes

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{1}{2} + X\right) \left(1 - \frac{1}{2} - X - \frac{1}{2} - Y\right) = -\frac{X}{2} - \frac{Y}{2} + \dots$$
(44)

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{1}{2} + Y\right) \left(\frac{3}{4} - \frac{1}{2} - Y - \frac{1}{4} - \frac{X}{2}\right) = -\frac{X}{4} - \frac{Y}{2} + \dots$$
(45)

Seeking a solution of the form $(X, Y) = \mathbf{a}e^{\lambda t}$, requires that

$$\begin{vmatrix} -1/2 - \lambda & -1/2 \\ -1/4 & -1/2 - \lambda \end{vmatrix} = 0, \quad \text{which implies} \quad \lambda = -\frac{1}{2} \pm \frac{1}{\sqrt{8}}.$$

This fixed point is therefore a stable node. The eigenvectors are given by: when $\lambda = -\frac{1}{2} + \frac{1}{\sqrt{8}}$, $\begin{pmatrix} -1/\sqrt{8} & -1/2 \\ -1/4 & -1/\sqrt{8} \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$. when $\lambda = -\frac{1}{2} - \frac{1}{\sqrt{8}}$, $\begin{pmatrix} 1/\sqrt{8} & -1/2 \\ -1/4 & 1/\sqrt{8} \end{pmatrix}$ $\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$. The general solution close to $(\frac{1}{2}, \frac{1}{2})$ is therefore

$$\begin{pmatrix} x - \frac{1}{2} \\ y - \frac{1}{2} \end{pmatrix} = c_7 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{(-1/2 + 1/\sqrt{8})t} + c_6 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{(-1/2 - 1/\sqrt{8})t}.$$
 (46)

These deductions characterise the behaviour of trajectories close to each of the fixed plane. Thus we may sketch the phase plane (see figure 6). All trajectories lead to $(x, y) = (\frac{1}{2}, \frac{1}{2})$ and thus this is the long time solution to the system, with the two populations co-existing.



Figure 6: The phase plane for the model of competing populations. The solid lines show some trajectories; the arrows show the vector field (\dot{x}, \dot{y}) .