Calculus (MATH11007): Sheet 10 2012

Second-order differential equations

1. For each of the following pairs of functions, show that the two functions are linearly independent:

   (a) $\{\cos(x), \sin(x)\}$

   (b) $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$, $\lambda_1 \neq \lambda_2$

   (c) $\{e^{\lambda x}, xe^{\lambda x}\}$

2. The angle, θ, that an undamped pendulum moving in a plane makes with the vertical is well described by

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta,$$

   for sufficiently small θ. The constants g and l are the acceleration due to gravity and the length of the pendulum, respectively. Find the general solution and hence the solutions satisfying the following boundary conditions:

   (a) $\theta(0) = a > 0$, $\theta'(0) = 0$;

   (b) $\theta(0) = 0$, $\theta'(0) = b > 0$;

   (c) $\theta(0) = a > 0$, $\theta'(0) = b > 0$. In this latter case write the solution in the form $\theta(t) = C\sin(\alpha t + \beta)$ and hence find the maximum value of θ.

3. The variable $u(t)$ is said to undergo damped harmonic motion if it satisfies

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}u}{\mathrm{d}t} + ku = 0,$$

   where the constants m, γ, k are all positive. Find the general solution in the cases: (a) $\gamma^2 > 4km$, (b) $\gamma^2 = 4km$ and (c) $\gamma^2 < 4km$. For each solution, determines what happens to $u(t)$ as $t \to \infty$.

4. Use the method of variation of parameters to find the general solution to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = f(x),$$

   when (a) $f(x) = x^2$ and (b) $f(x) = \cot x$.

5. (a) Show that if $x = \mathrm{e}^s$ then $x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s}$.

   (b) Hence find the general solution of

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 0.$$