Calculus (MATH11007): Sheet 14 — 2013

Coupled first order systems II

1. Consider the following pair of coupled equations:

$$\dot{x} = 4y, \quad \dot{y} = -x. \tag{1}$$

In this question you will solve these equations in three different ways.

(a) Find the general solution by using a trial solution of the form

$$\mathbf{x} \equiv \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathbf{a}\mathrm{e}^{\lambda t}.$$

(b) By differentiating the second equation in (1) with respect to t, and substituting for $\dot{x}$ find a second order linear equation for y. Hence find the general solution for $y(t)$ and therefore deduce the solution for $x(t)$.

(c) Use the differential equations (1) to find an equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$. Solve this equation to find the trajectories. Also use this solution to derive an expression for $x(t)$, by substituting into the first equation in (1). Confirm that your answer agrees with your solution from the previous parts of the question.

2. Find the general solutions of the following coupled first order systems

(a)
$$\dot{x} = x + 4y, \qquad \dot{y} = -x + y.$$

(b)
$$\dot{x} = -x + 4y, \qquad \dot{y} = -x - y.$$

In both cases sketch a typical trajectory in the phase plane.

3. Consider the following coupled equations for variables $x \geq 0$ and $y \geq 0$:

$$\begin{aligned} \dot{x} &= 4x - 4xy, \\ \dot{y} &= -9y + 18xy. \end{aligned} \tag{2}$$

(a) Find the equilibrium points and by expanding around them, find the form of the linearized solutions near to the equilibrium points.

(b) Find equations for the trajectories in the phase plane near to the equilibrium points.

(c) Use (2) to find an equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$, and hence the exact trajectories throughout the phase plane. Show that near to the equilibrium points identified in (a), the from of the trajectories reduces to what you found in part (b) .

4. The populations of competing species are denoted by $x(t)$ and $y(t)$ and satisfy the following coupled differential equations

$$\begin{aligned} \dot{x} &= x\left(1 - x - y\right), \\ \dot{y} &= y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right). \end{aligned}$$

(a) Find the equilibrium points of this coupled system and by linearising about them classify each of them.

(b) Sketch the trajectories of system in the phase plane. What is the solution as $t \to \infty$?