Calculus (MATH11007): Sheet 7 — 2012

First-order differential equations

1. Solve the following. If no initial condition is given, find the general solution; if an initial condition is given, find the largest region of validity of the solution.

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = \mathrm{e}^x\,, \quad y(0) = 1.$

   (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{y}.$

   (c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{2y}{x} = x^2.$

   (d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2\,, \quad y(0) = 1.$

   (e) $\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{tx}{t^2+1} = t\,, \quad x(0) = 1.$

   (f) $\dfrac{\mathrm{d}s}{\mathrm{d}t} = s^2 \sin t\,, \quad s(0) = 2.$

   (g) $\dfrac{\mathrm{d}p}{\mathrm{d}s} = \dfrac{sp}{s^2+1}.$

   (h) $u' + u\cos t = \cos t.$

   (i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2x^3.$

   (j) $(2t + x)\dfrac{\mathrm{d}x}{\mathrm{d}t} + t = 0.$

2. Find all the solutions of the following differential equations.

   (a) $(1 - x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{4 - y^2}\,.$

   (b) $y' + ay = b\,.$

   (c) $(y')^2 - 3y' = -2\,, \quad y(1) = 2$

   In (b), consider all possible values of the constants a and b.

3. The Bernoulli differential equation may be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + g(x)y^{\nu}\,,$$

   where f and g are given functions and $\nu \neq 1$ is a given real number. Show that the substitution $y(x) = [u(x)]^{\alpha}$, where α is to be determined, transforms the equation into a linear equation equation which may then be solved analytically. Use this idea to solve the following:

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + xy = xy^3\,, \quad y(0) = \dfrac{1}{\sqrt{2}}\,.$

   (b) $\dfrac{\mathrm{d}\sigma}{\mathrm{d}t} - \sigma = \dfrac{t}{\sigma}\,, \quad \sigma(0) = 1\,.$

4. Find all the solutions of the equation

$$(1 - x^2)\,(y')^2 = 1 - y^2\,.$$

5. The settling velocity of a small sphere in water, $u(t)$, satisfies

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\Delta\rho g}{\rho_s} - \frac{18\mu u}{\rho_s d^2},$$

where ρ_s denotes the density of the sphere, $\Delta\rho$ denotes the density difference between the sphere and water, g denotes gravitational acceleration, d denotes the diameter of the sphere and μ denotes the viscosity of water.

(a) Find the velocity, $u(t)$, given that the sphere starts from rest ($u(0) = 0$).

(b) Calculate the terminal settling velocity $V_s = \lim_{t\to\infty} u(t)$.

(c) Find the time, t_s, when the velocity equals $(1 - e^{-1})V_s$.

(d) Evaluate t_s for a 100μm particle of sand ($\rho_s = 2600\text{Kgm}^{-3}$) settling through water of density $\rho = 1000\text{Kgm}^{-3}$ and viscosity $\mu = 10^{-3}\text{Kgm}^{-1}\text{s}^{-1}$.

6. The growth rate of the mass of material, $x(t)$, in a chemical reaction satisfies the following the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = K(a - x)(b - x)\,, \quad x(0) = 0\,,$$

where K, a and b are positive constants such that $a \neq b$. Solve the equation and deduce that

$$\lim_{t\to\infty} x(t) = \begin{cases} a, & \text{if } a \leq b, \\ b, & \text{if } b < a. \end{cases}$$