Calculus (MATH11007): Sheet 8 **2012**

First-order differential equations

1. The population, $P(t)$, of a biological species can be modelled by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(\beta - \delta P), \qquad \text{subject to} \qquad P(0) = P_0,$$

where β is the "birth rate" and δ is the "death rate". For example, this might model the population of fish in the ocean.

Assuming $P > 0$, find the population $P(t)$ for the three cases: (i) $P_0 < \hat{P}$, (ii) $P_0 = \hat{P}$ and (iii) $P_0 > \hat{P}$, where $\hat{P} = \beta/\delta$. Sketch the solutions.

2. Solve the following

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{y^2 + x}.$$

[Hint: consider making x the dependent variable]

3. Given that $y(x)$ satisfies the following

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{\sin(x)}; \qquad y(0) = 0,$$

show that

$$\frac{x}{c} \le y(x) \le xc \qquad \forall x \ge 0,$$

where c is a constant that you should determine.

4. Solve the following equations; in both cases state the region for which your solution is valid and sketch the solution.

(a) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -\dfrac{x^3}{2}; \qquad x(0) = 1.$ (b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{x^3}{2}; \qquad x(0) = 1.$

5. (a) Solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{x}; \qquad x(0) = 1.$$

(b) Use separation of variables to solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{x}; \qquad x(0) = 0.$$

(c) Show that $x(t) = 0$ is also a solution to part (b).

Why is there a unique solution to (a), but a non-unique solution to (b)?

6. Solve the following equations; in both cases state the region for which your solution is valid and sketch the solution.

   (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 - y - y^2; \qquad y(0) = 0.$ (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 - y - y^2; \qquad y(0) = 4.$

7. (a) Find the solutions to the following equation if they exists:

$$x\frac{\mathrm{d}\phi}{\mathrm{d}x} + \phi = x\cos x,$$

   subject to (i) $\phi(0) = 0$ and (ii) $\phi(0) = 1$

   (b) Comment on what you find in the context of the existence theorem for ordinary differential equations.

8. Consider the following equation and initial condition

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, \qquad y(0) = 1.$$

   (a) Find the solution $y(x)$ and evaluate $y(a)$.

   (b) Now consider a numerical approximation to the solution, constructed using Euler's method, with a step size h so that $x_n = nh$.

   (c) Deduce that $y_n = (1 + h)^n$.

   (d) Write down the numerical approximation to y at $x = a$ using Euler's method with N equal sized steps and show that this numerical solution converges to the exact value as $N \to \infty$.

9. Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3 + e^{-t} - \frac{y}{2}; \qquad y(0) = 1.$$

   (a) Solve the equation analytically.

   (b) Use `Matlab` to compute a numerical approximate solution for $y(t)$ using Euler's method for t between $t = 0$ and $t = 3$. Do this for time steps of size (i) 1, (ii) 0.1, (iii) 0.01.

   (c) Use `Matlab` to plot all four solutions (the analytical one and all three approximations) on the same graph.