Calculus (MATH11007): Solutions 10 — 2012

Second-order differential equations

1. (a) We seek a and b (non zero) such that $a\cos x + b\sin x = 0$ for all x.
   Choose $x = 0$ and this implies $a = 0$. Then choose $x = \pi/2$ and this implies $b = 0$. Hence the only solution is $a = b = 0$ and thus $\{\sin x, \cos x\}$ are independent.

   (b) We seek a and b (non zero) such that $a\mathrm{e}^{\lambda_1 x} + b\mathrm{e}^{\lambda_2 x} = 0$ for all x.
   Choose $x = 0$ and this implies $a + b = 0$. Then choose $x = 1$ and this implies $a(\mathrm{e}^{\lambda_1} - \mathrm{e}^{\lambda_2}) = 0$, which leads to $a = b = 0$ if $\lambda_1 \neq \lambda_2$. Hence $\{\mathrm{e}^{\lambda_1 x}, \mathrm{e}^{\lambda_2 x}\}$ are independent.

   (c) We seek a and b (non zero) such that $a\mathrm{e}^{\lambda_1 x} + bx\mathrm{e}^{\lambda_2 x} = 0$ for all x.
   Choose $x = 0$ and this implies $a = 0$. Then choose $x = 1$ and this implies $b\mathrm{e}^{\lambda_1} = 0$, which leads to $b = 0$. Hence $\{\mathrm{e}^{\lambda_1 x}, x\mathrm{e}^{\lambda_1 x}\}$ are independent.

2. We seek a solution of the form $\theta(t) = \mathrm{e}^{rt}$, which gives

$$r^2 + \omega^2 = (r - \mathrm{i}\omega)(r + \mathrm{i}\omega) = 0,$$

where $\omega^2 = g/l$. Thus $r = \pm\mathrm{i}\omega$ and the general solution is $\theta(t) = A\cos\omega t + B\sin\omega t$.

   (a) When $\theta(0) = a$ and $\theta'(0) = 0$, the solution is $\theta(t) = a\cos\omega t$.

   (b) When $\theta(0) = 0$ and $\theta'(0) = b$, the solution is $\theta(t) = \dfrac{b}{\omega}\sin\omega t$.

   (c) When $\theta(0) = a$ and $\theta'(0) = b$, the solution is

$$\begin{aligned}\theta(t) &= a\cos\omega t + \frac{b}{\omega}\sin\omega t \\ &= \left(a^2 + \frac{b^2}{\omega^2}\right)^{1/2} \sin\left(\omega t + \tan^{-1}\left(\frac{a\omega}{b}\right)\right).\end{aligned}$$

   Thus $\max(\theta(t)) = (a^2 + b^2/\omega^2)^{1/2}$.

3. Seeking a solution of the form $u(t) = \mathrm{e}^{rt}$ we find that

$$mr^2 + \gamma r + k = 0.$$

This has solutions $r = (-\gamma \pm \sqrt{\gamma^2 - 4km})/(2m)$. The nature of the solution depends on whether $\Delta^2 = \gamma^2 - 4km$ is positive or negative.

   (a) If $\Delta^2 > 0$, then the two roots are both real and given by

$$r_1 = \frac{-\gamma + \Delta}{2m} \qquad \text{and} \qquad r_2 = \frac{-\gamma - \Delta}{2m}.$$

   The general solution is $u(t) = A\mathrm{e}^{r_1 t} + B\mathrm{e}^{r_2 t}$. Both $r_1 < 0$ and $r_2 < 0$ and so $u(t) \to 0$ as $t \to \infty$.

   (b) If $\Delta^2 = 0$, then the root $r = -\gamma/(2m)$ is repeated and the general solution is given by $u(t) = (A + Bt)\mathrm{e}^{rt}$. We find that $u(t) \to 0$ as $t \to \infty$.

(c) If $\Delta^2 = -\delta^2 < 0$ then the two roots are complex conjugates given by

$$r_1 = \frac{-\gamma + \mathrm{i}\delta}{2m} \qquad \text{and} \qquad r_2 = \frac{-\gamma - \mathrm{i}\delta}{2m}.$$

The general solution is $u(t) = \mathrm{e}^{-\gamma t/(2m)}\left(A\sin\frac{\delta t}{2m} + B\cos\frac{\delta t}{2m}\right)$. Thus $u(t) \to 0$ as $t \to \infty$.

4. The general solution to the homogeneous problem is $y(x) = A\sin x + B\cos x$. Thus using the method of variation of parameters to final a particular integral we write $y(x) = c(x)\sin x + d(x)\cos x$.

Evaluating the derivative $y' = c'\sin x + c\cos x + d'\cos x - d\sin x$. We choose to enforce $c'\sin x + d'\cos x = 0$.

Evaluating the second derivative $y'' = c'\cos x - c\sin x - d'\sin x - d\cos x$. Thus $y'' + y = f(x)$ yields $c'\cos x - d'\sin x = f(x)$

The differential equations for $c(x)$ and $d(x)$ are then

$$c' = f(x)\cos x \qquad \text{and} \qquad d' = -f(x)\sin x$$

(a) When $f(x) = x^2$, we find $d(x) = x^2\cos x - 2x\sin x - 2\cos x$ and $c(x) = x^2\sin x - 2\sin x + 2x\cos x$. (Here the integration constants can be neglected because their contributions are already in the complementary function.) Putting the solutions together we find that

$$y(x) = x^2 - 2 + A\sin x + B\cos x.$$

(b) When $f(x) = \cot x$, we find $d(x) = -\sin x$ and $c(x) = \cos x - \ln(\mathrm{cosec} x - \cot x)$. (Again, the integration constants can be neglected because their contributions are already in the complementary function.) Putting the solutions together we find that

$$y(x) = \sin x \ln\left(\frac{\sin x}{1 - \cos x}\right) + A\sin x + B\cos x.$$

5. (a) If $x = \mathrm{e}^s$ then $\mathrm{d}x/\mathrm{d}s = \mathrm{e}^s = x$. So $x\mathrm{d}y/\mathrm{d}x = x\mathrm{d}y/\mathrm{d}s(\mathrm{d}s/\mathrm{d}x) = \mathrm{d}y/\mathrm{d}s$.

(b) Note that $\frac{\mathrm{d}^2y}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x}$. Then the differential equation transforms to

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - 4y = 0.$$

This may be solved by looking for solutions of the form $y = \mathrm{e}^{rs}$, for which we deduce $r^2 - 4 = 0$ and so $r = \pm 2$. Then the solution is given by

$$y(x) = A\mathrm{e}^{2\ln(x)} + B\mathrm{e}^{-2\ln(x)} = Ax^2 + \frac{B}{x^2}.$$