Calculus (MATH11007): Solutions 11 — 2013
Discrete Dynamical Systems I

1. (a) If $u(n) = E$, then $E = \frac{1}{2}E + \frac{3}{2}$ and so $E = 3$.
Writing $u(n) = 3 + \delta_n$, we find that $3 + \delta_{n+1} = \frac{1}{2}(3 + \delta_n) + \frac{3}{2}$. Thus

$$\delta_{n+1} = \frac{1}{2}\delta_n,$$

and so $\delta_n \to 0$ as $n \to \infty$. Thus the equilibrium point $u(n) = 3$ is stable.

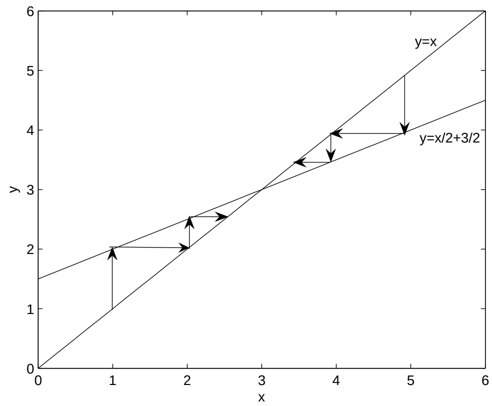

Figure 1: Web plot for $u(n+1) = \frac{1}{2}u(n) + \frac{3}{2}$, with (i) $u(0) = 1$; and (ii) $u(0) = 5$.

(b) If $u(n) = E$, then $E = 2E + 2$ and so $E = -2$.
Writing $u(n) = -2 + \delta_n$, we find that $-2 + \delta_{n+1} = 2(-2 + \delta_n) + 2$. Thus

$$\delta_{n+1} = -2\delta_n,$$

and so $|\delta_n| \to \infty$ as $n \to \infty$. Thus the equilibrium point $u(n) = -2$ is unstable.

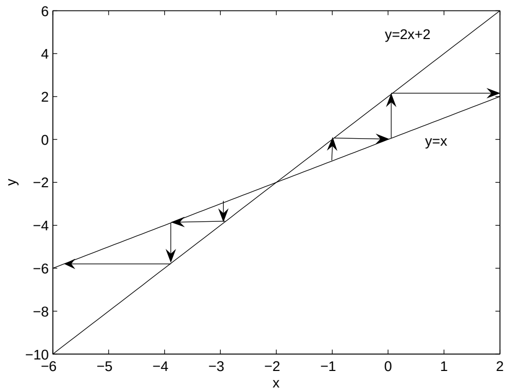

Figure 2: Web plot for $u(n+1) = 2u(n) + 2$, with (i) $u(0) = -1$; and (ii) $u(0) = -3$.

(c) If $u(n) = E$, then $E = -E + 2$ and so $E = 1$.
Writing $u(n) = 1 + \delta_n$, we find that $1 + \delta_{n+1} = -(1 + \delta_n) + 2$. Thus

$$\delta_{n+1} = -\delta_n,$$

and so $\delta_{n+2} = \delta_n$. Thus the difference between $u(n)$ and the equilibrium point does not change.

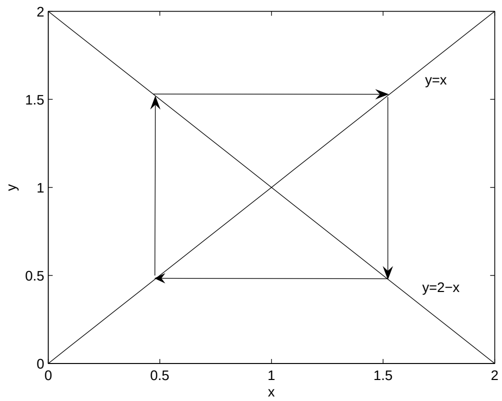

Figure 3: Web plot for $u(n + 1) = -u(n) + 2$, with $u(0) = 1.5$.

2. The method is to find a complimentary function and a particular integral.
The complimentary function: solution to the homogeneous problem $u(n+1) = 2u(n)$. Try $u(n) = \lambda^n$ and so $\lambda^{n+1} = 2\lambda^n$. Thus $\lambda = 2$ and

$$u(n) = A2^n, \qquad \text{with } A \text{ an arbitrary constant.}$$

The particular integral: try $u(n) = c$ and so $c = 2c + 3$. Thus $c = -3$.

The general solution is $u(n) = A2^n - 3$.

To satisfy $u(0) = 3$, we require $3 = A - 3$ and so the solution is

$$u(n) = 6\ 2^n - 3 = 3(2^{n+1} - 1).$$

3. The method is to find a complimentary function and a particular integral.
The complimentary function: solution to the homogeneous problem $u(n + 1) = -2u(n)$. Try $u(n) = \lambda^n$ and so $\lambda^{n+1} = -2\lambda^n$. Thus $\lambda = -2$

$$u(n) = A(-2)^n, \qquad \text{with } A \text{ an arbitrary constant.}$$

The particular integral: try $u(n) = c$ and so $c = -2c + 3$. Thus $c = 1$.

The general solution is $u(n) = A(-2)^n + 1$.

To satisfy $u(0) = 3$, we require $3 = A + 1$ and so the solution is

$$u(n) = 2(-2)^n + 1.$$

4. (a) Let $u(n)$ be the amount of drug in the patient's blood on day n and let d be the daily dose. Then

$$u(n + 1) = \frac{3}{5}u(n) + d.$$

The required equilibrium value $u(n) = E = 40$mg and so since $E = \frac{3}{5}E + d$, we deduce that $d = 2E/5 = 16$mg.

(b) The general solution is $u(n) = A\left(\frac{3}{5}\right)^n + 40$, where A is a constant. Thus the equilibrium value is stable.
If $u(0) = 80$mg, then because the equilibrium value is approached monotonically from above, the dangerous amount (100mg) is never exceeded.

(c) If $u(0) = 200$, then $u(1) = 136$ and $u(2) = 97.6$. So the dangerous limit is exceeded for under 2 days.

5. (a) If $u(n) = E$ then $E = -E$ and so $E = 0$. There are no non-zero equilibrium points.

(b) $u(n+2) = -u(n+1) = u(n)$ and so all values are fixed points of order 2.

(c) One possibility is that $u(n+1) = \omega u(n)$ and so if there is a fixed point of order 3, $\omega^3 = 1$ (with $\omega \neq \pm 1$ as the map does not have fixed points of order 1 or 2). Thus the map is

$$u(n+1) = \exp\left(\frac{2\pi \mathrm{i}}{3}\right) u(n).$$

A generalisation to maps with fixed points of order M, but no fixed points of order $m < M$, is given by

$$u(n+1) = \exp\left(\frac{2\pi \mathrm{i}}{M}\right) u(n).$$