Calculus (MATH11007): Solutions 12 **2013**

Discrete Dynamical Systems II

1. (a) (i) $u(0) = 1$, $u(1) = 2$, $u(2) = -3$, $u(3) = 1$, $u(4) = 2$, $u(5) = -3$.
   (ii) $u(0) = 0$, $u(1) = 2$, $u(2) = -2$, $u(3) = 0$, $u(4) = 2$, $u(5) = -2$.

   (b) Seek a solution of the form $u(n) = A\lambda^n$, so $\lambda^2 + \lambda + 1 = 0$. Thus $\lambda = (-1 \pm \mathrm{i}\sqrt{3})/2 = \exp(\pm \mathrm{i}2\pi/3)$ and so

   $$u(n) = A\exp\left(\mathrm{i}n2\pi/3\right) + B\exp\left(-\mathrm{i}n2\pi/3\right).$$

   Then

   $$u(n+3) = A\exp\left(\mathrm{i}(n+3)2\pi/3\right) + B\exp\left(-\mathrm{i}(n+3)2\pi/3\right) = u(n),$$

   which implies that that the series is periodic with period 3.

   (c) Seek a solution of the form $u(n) = A\lambda^n$, so $\lambda^2 - \lambda + 1 = 0$. Thus $\lambda = (1 \pm \mathrm{i}\sqrt{3})/2 = \exp(\pm \mathrm{i}\pi/3)$ and so

   $$u(n) = A\exp\left(\mathrm{i}n\pi/3\right) + B\exp\left(-\mathrm{i}n\pi/3\right).$$

   From this expression, we deduce that $u(n+6) = u(n)$ and so the series is periodic with period 6.

2. We construct the solution by finding the complimentary function and a particular integral. The complimentary function: if $u(n+1) = \frac{1}{2}u(n)$ then

   $$u(n) = A\left(\frac{1}{2}\right)^n, \qquad \text{with } A \text{ an arbitrary constant.}$$

   For the particular integral, try $u(n) = an + b$, so $a(n+1) + b = \frac{1}{2}(an+b) + n$. This implies that $n(a/2 - 1) + a + b/2 = 0$ and thus $a = 2$ and $b = -4$. The general solution is given by

   $$u(n) = A\left(\frac{1}{2}\right)^n + 2n - 4.$$

   If $u(0) = 0$, then $A = 4$ and

   $$u(n) = 4\left(\frac{1}{2}\right)^n + 2n - 4.$$

3. (a) Sketch for web-plots:

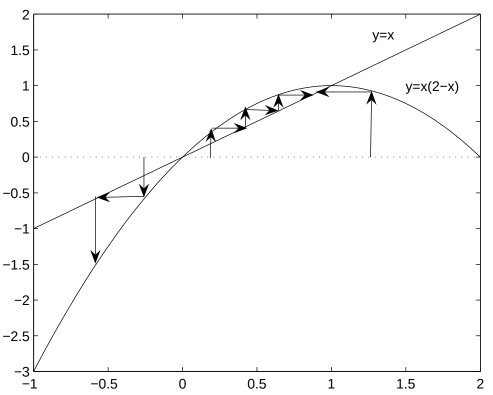

Figure 1: Web plots for the map $u(n+1) = u(n)(2 - u(n))$.

(b) The equilibrium points satisfy $E = E(2 - E)$ and so $E = 0$ or $E = 1$.
First we examine $E = 0$ and substitute $u(n) = \delta_n$. Thus

$$\delta_{n+1} = \delta_n(2 - \delta_n) = 2\delta_n + \dots$$

So $\delta_n = A2^n$ and the equilibrium point $u(n) = 0$ is unstable.
Next we examine $E = 1$ and substitute $u(n) = 1 + \delta_n$. Thus

$$1 + \delta_{n+1} = (1 + \delta_n)(2 - 1 - \delta_n) = 1 - \delta_n^2$$

So if $|\delta_n| < 1$ then $|\delta_{n+1}| < |\delta_n|$ and the equilibrium point $u(n) = 1$ is stable.
Hence we deduce that for initial points in range (i) if $u(0) < 0$ then $u(n) \to -\infty$ as $n \to \infty$; and for initial points in ranges (ii) and (iii) if $0 < u(0) < 2$ then $u(n) \to 1$ as $n \to \infty$.

4. (a) If $u(n) = E$ then $E = \frac{7}{2}E(1 - E)$ and so the fixed points of order one are $E = 0$ and $E = 5/7$.

(b) If $u(2n) = E_1$ and $u(2n+1) = E_2$ then

$$E_2 = \frac{7}{2}E_1(1 - E_1) \qquad \text{and} \qquad E_1 = \frac{7}{2}E_2(1 - E_2).$$

Eliminating E_2, we find that

$$E_1 - \frac{49}{4}E_1(1 - E_1)\left(1 - \frac{7}{2}E_1(1 - E_1)\right) = 0.$$

This may be factorised to give

$$\frac{1}{8}E_1(7E_1 - 5)(7E_1 - 6)(7E_1 - 3) = 0,$$

with solutions (i) $E_1 = E_2 = 0$; (ii) $E_1 = E_2 = 5/7$; (iii) $E_1 = 6/7$, $E_2 = 3/7$; and (iv) $E_1 = 3/7$, $E_2 = 6/7$.

(c) Examining the stability:
(i) $u(0) = \delta$ and so $u(1) = \frac{7}{2}\delta(1-\delta) = \frac{7}{2}\delta + \ldots$. Thus this fixed point is unstable.
(ii) $u(0) = \frac{5}{7} + \delta$ and so $u(1) = \frac{7}{2}(\frac{5}{7}+\delta)(1-\frac{5}{7}-\delta) = \frac{5}{7} - \frac{3}{2}\delta + \ldots$. Thus this fixed point is unstable.
(iii) $u(0) = \frac{3}{7} + \delta$, so $u(1) = \frac{7}{2}(\frac{5}{7}+\delta)(1-\frac{5}{7}-\delta) = \frac{6}{7} + \frac{1}{2}\delta + \ldots$ and then $u(2) = \frac{7}{2}(\frac{6}{7}+\frac{1}{2}\delta)(\frac{1}{7}-\frac{1}{2}\delta) = \frac{3}{7} - \frac{5}{4}\delta + \ldots$. Thus this fixed point of period 2 is unstable.

5. The figures of the numerical output:

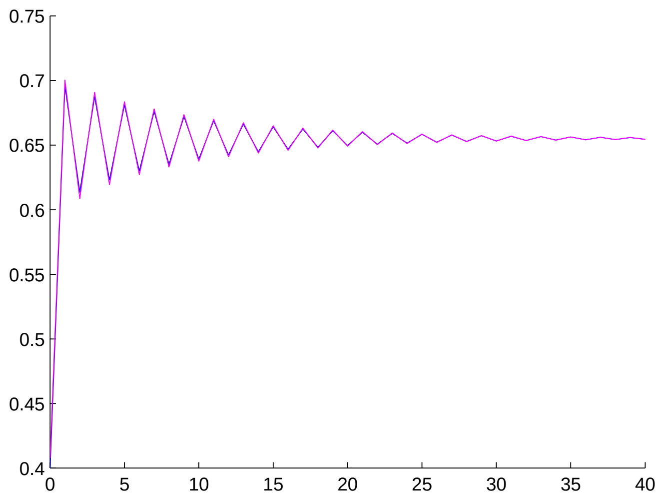

Figure 2: The logistic map with $a = 2.9$. There is a stable fixed point of period 1 ($u(n) = 0.655$).

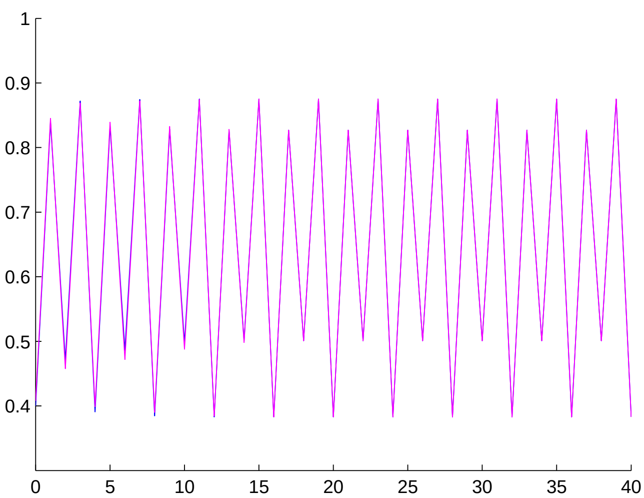

Figure 3: The logistic map with $a = 3.5$. There is a stable fixed point of period 4 ($u(4n) = 0.383$, $u(4n+1) = 0.827$, $u(4n+2) = 0.501$, $u(4n+3) = 0.875$).

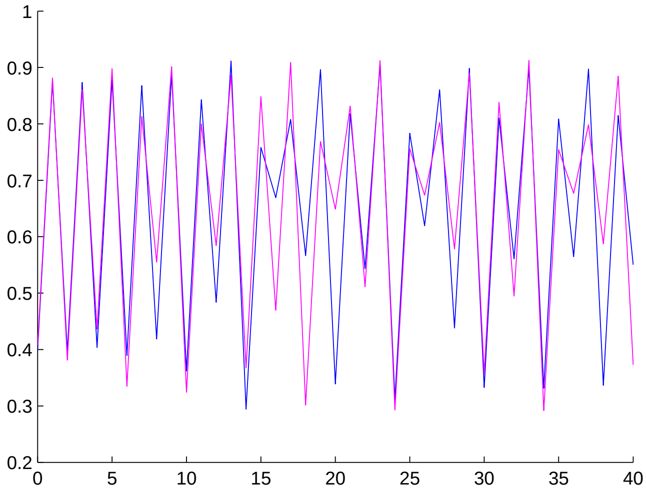

Figure 4: The logistic map with $a = 3.65$.

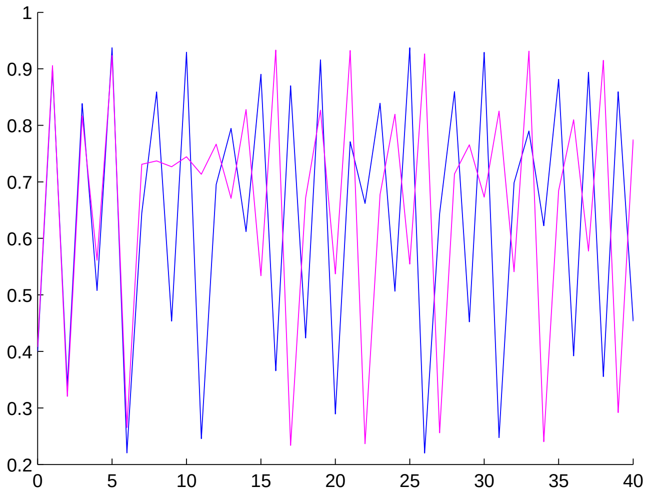

Figure 5: The logistic map with $a = 3.75$.