Calculus (MATH11007): Solutions 13 **2013**

Coupled first order systems

1. (a) Solve $3x + 2y = 0$ and $x = 0$ to find the equilibrium point is $(x, y) = (0, 0)$.
   (b) Solve $3x+2y+1 = 0$ and $x+2y+4 = 0$ to find the equilibrium point is $(x, y) = (\frac{3}{2}, -\frac{11}{4})$.
   (c) Solve $3x + 6y = 0$ and $2x + 4y = 0$ to find the equilibrium points lie along the line $x = -2y$ and are given parametrically by $(x, y) = (t, -\frac{1}{2}t)$ for all values of the parameter t.
   (d) Solve $x(1-y) = 0$ and $x-3y+2xy = 0$ to find the equilibrium points are $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$.
   (e) Solve $v = 0$ and $x - x^5 = 0$ to find the equilibrium points are $(x, v) = (0, 0)$ and $(x, v) = (\pm 1, 0)$.
   (f) Solve $v = 0$ and $x + x^5 = 0$ to find the equilibrium point is $(x, v) = (0, 0)$.

2. (a) The coupled equations may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Seek a solution of the form $\mathbf{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{a}\mathrm{e}^{\lambda t}$. This requires

$$\left[\begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{a} = 0.$$

A non-trivial solution is admissible if $\begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = 0$. Thus $(\lambda+2)(\lambda-7)-36 = 0$, which factorises to $(\lambda - 10)(\lambda + 5) = 0$. Hence there are two distinct values for λ, namely $\lambda = -5$ and $\lambda - 10$.

When $\lambda = 10$, $\begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix}\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

When $\lambda = -5$, $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The general solution is then

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}\mathrm{e}^{10t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}\mathrm{e}^{-5t}, \qquad (1)$$

where c_1 and c_2 are constants.

   (b) The phase plane is given in figure 1
   (c) At $t = 0$, $(x, y) = (1, 0)$ and so from (1),

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus $c_1 = 1/5$ and $c_2 = -2/5$ and the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\mathrm{e}^{10t} - \frac{2}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\mathrm{e}^{-5t},$$

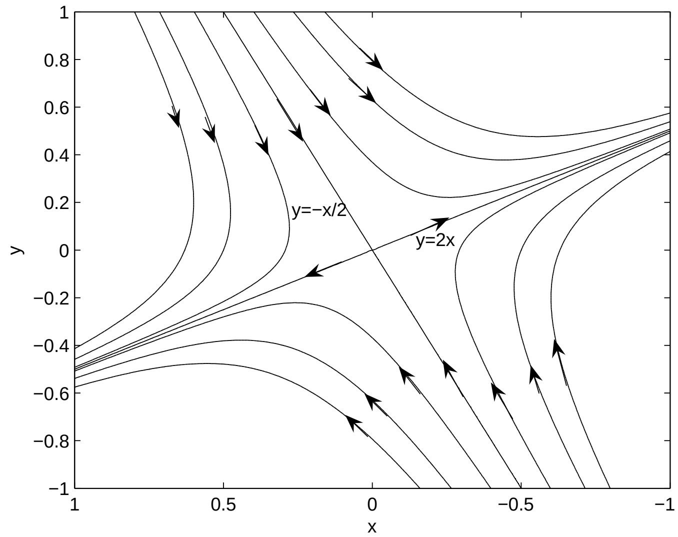

Figure 1: The phase plane for question 2(b).

3. (a) The coupled equations may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Seek a solution of the form $\mathbf{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{a}\mathrm{e}^{\lambda t}$. This requires

$$\left[\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{a} = 0.$$

A non-trivial solution is admissible if $\begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = 0$. Thus $(\lambda-6)(\lambda-9)-4=0$, which factorises to $(\lambda-10)(\lambda-5)=0$. Hence there are two distinct values for λ, namely $\lambda = 5$ and $\lambda = 10$.

When $\lambda = 10$, $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

When $\lambda = 5$, $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The general solution is then

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}\mathrm{e}^{10t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}\mathrm{e}^{5t}, \tag{2}$$

where c_1 and c_2 are constants.

(b) The phase plane is given in figure 2

(c) At $t = 0$, $(x, y) = (1, 0)$ and so from (2),

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus $c_1 = 1/5$ and $c_2 = -2/5$ and the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\mathrm{e}^{10t} - \frac{2}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\mathrm{e}^{-5t},$$

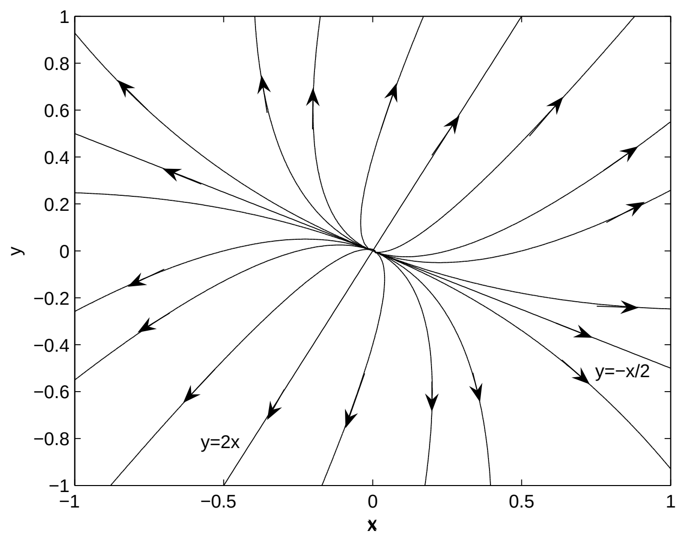

Figure 2: The phase plane for question 3(b).

4. (a) The coupled equations may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Seek a solution of the form $\mathbf{x} \equiv \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{v}\mathrm{e}^{\lambda t}$. This requires

$$\left[\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{v} = 0.$$

A non-trivial solution is admissible if $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$. Thus $(\lambda-1)(\lambda-3)+1 = 0$, which factorises to $(\lambda-2)^2 = 0$. Hence there is one two distinct value for λ, namely $\lambda = 2$.

When $\lambda = 2$, $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{v} = 0$ and so $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) Now seek an additional solution of the form $\mathbf{x} = \mathbf{w}t\mathrm{e}^{2t} + \mathbf{z}\mathrm{e}^{2t}$. The derivative is then given by $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \equiv \mathbf{w}\,(1+2t)\,\mathrm{e}^{2t} + 2\mathbf{z}\mathrm{e}^{2t}$ and the governing equation is then of the form

$$\mathbf{w}\,(1+2t)\,\mathrm{e}^{2t} + 2\mathbf{z}\mathrm{e}^{2t} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\left(\mathbf{w}t\mathrm{e}^{2t} + \mathbf{z}\mathrm{e}^{2t}\right).$$

Equating the terms in e^{2t} and $t\mathrm{e}^{2t}$, we deduce

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{w} = 0 \quad \text{and} \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{z} = \mathbf{w}.$$

So we find that $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} -1-b \\ b \end{pmatrix}$, for any b.

So the general solution can be written

$$\mathbf{x} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathrm{e}^{2t} + c_2\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}t\mathrm{e}^{2t} + \begin{pmatrix} -1-b \\ b \end{pmatrix}\mathrm{e}^{2t}\right),$$

where c_1 and c_2 are constants. Then by writing $c_3 = c_1 - bc_2$, we find

$$\mathbf{x} = c_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{2t} + c_2 \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} t\mathrm{e}^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathrm{e}^{2t} \right).$$