Calculus (MATH11007): Solutions 14 — 2013

Coupled first order systems II

1. (a) Substituting $\mathbf{x} = \mathbf{a}\mathrm{e}^{\lambda t}$, we find the equations are satisfied if

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{a} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \mathbf{a}.$$

This has a non-trivial solution if $\begin{vmatrix} -\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = 0$. Thus $\lambda = \pm 2\mathrm{i}$.

When $\lambda = 2\mathrm{i}$, $\begin{pmatrix} -2\mathrm{i} & 4 \\ -1 & -2\mathrm{i} \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} -2\mathrm{i} \\ 1 \end{pmatrix}$.

When $\lambda = -2\mathrm{i}$, $\begin{pmatrix} 2\mathrm{i} & 4 \\ -1 & 2\mathrm{i} \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 2\mathrm{i} \\ 1 \end{pmatrix}$.

So the general solution is given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -2\mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{2\mathrm{i}t} + c_2 \begin{pmatrix} 2\mathrm{i} \\ 1 \end{pmatrix} \mathrm{e}^{-2\mathrm{i}t} = c_3 \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} + c_4 \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix},$$

where $c_3 = c_1 + c_2$ and $c_4 = \mathrm{i}(c_1 - c_2)$.

(b) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\dfrac{\mathrm{d}x}{\mathrm{d}t} = -4y$. We seek a solution to this second order linear differential equation of the form $y(t) = \mathrm{e}^{\lambda t}$ and so $\lambda^2 + 4 = 0$. Then the general solution is

$$y(t) = A\cos 2t + B\sin 2t,$$

where A and B are constants. Substituting to find $x(t)$ gives

$$x(t) = 2A\sin 2t - 2B\cos 2t.$$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = -\dfrac{x}{4y}$. Then integrating we find that $4y^2 + x^2 = R^2$, where R is a constant. This implies that trajectories in the phase plane are ellipses.

Next we substitute for y to find $\dot{x} = 2(R^2 - x^2)^{1/2}$. This first-order equation may be solved by separating variables to find that

$$\int \frac{\mathrm{d}x}{(R^2 - x^2)^{1/2}} = \sin^{-1}\left(\frac{x}{R}\right) = 2t + c_5.$$

Then we deduce that

$$x = R\sin(2t + c_5) = c_6 \cos 2t + c_7 \sin 2t,$$

where c_6 and c_7 are constants. Further substitution into $y = \dot{x}/4$ gives

$$y = -\frac{c_6}{2}\sin 2t + \frac{c_7}{2}\cos 2t.$$

This recovers the solutions from (a) and (b).

2. (a) We seek a solution of the form $\mathbf{x} = \mathbf{a}e^{\lambda t}$, which requires

$$\begin{vmatrix} 1-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0.$$

So $(1-\lambda)^2 + 4 = 0$ and this implies that $\lambda = 1 \pm 2i$.

When $\lambda = 1+2i$, $\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$.

When $\lambda = 1-2i$, $\begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$.

The general solution is then written as

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{2it} + c_2 e^t \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{-2it} = e^t \left[c_3 \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} + c_4 \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix} \right],$$

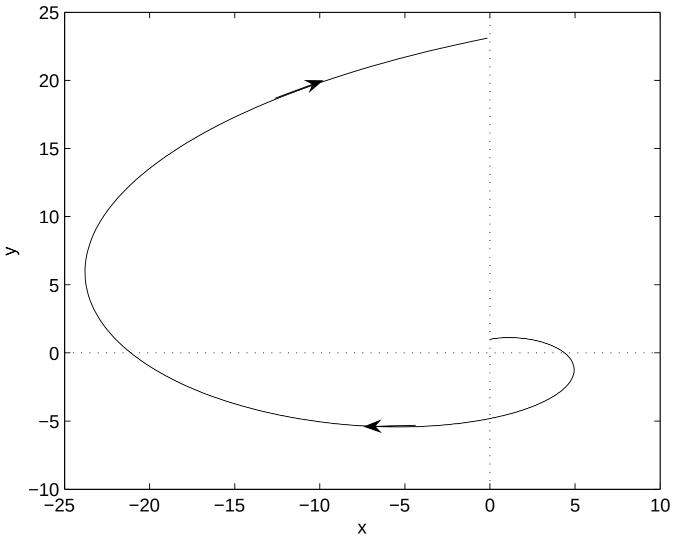

Figure 1: Trajectory for 2(a). A 'flattened' spiral.

(b) We seek a solution of the form $\mathbf{x} = \mathbf{a}e^{\lambda t}$, which requires

$$\begin{vmatrix} -1-\lambda & 4 \\ -1 & -1-\lambda \end{vmatrix} = 0.$$

So $(-1-\lambda)^2 + 4 = 0$ and this implies that $\lambda = -1 \pm 2i$.

When $\lambda = -1+2i$, $\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$.

When $\lambda = -1-2i$, $\begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$.

The general solution is then written as

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{2it} + c_2 e^{-t} \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{-2it} = e^{-t} \left[c_3 \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} + c_4 \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix} \right],$$

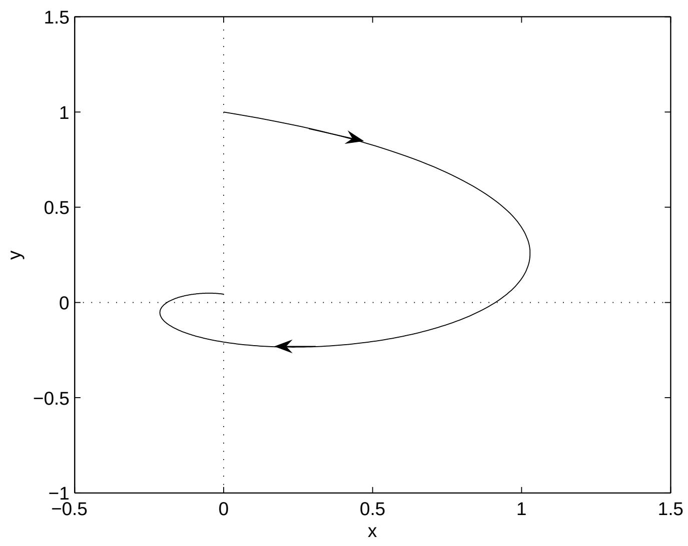

Figure 2: Trajectory for 2(b). A 'flattened' spiral.

3. (a) The equilibrium points are given by simultaneously solving $4x(1-y) = 0$ and $9y(-1+2x) = 0$, which yields $(x, y) = (0, 0)$ and $(x, y) = (\frac{1}{2}, 1)$.
We now linearise the equations about these equilibrium points.
(i) Close to $(0, 0)$, the equations become

$$\dot{x} = 4x + \dots \qquad \text{and} \qquad \dot{y} = -9y + \dots.$$

Thus $x(t) = A\mathrm{e}^{4t}$ and $y(t) = B\mathrm{e}^{-9t}$, where A and B are constants. This means that the origin is a 'saddle'.
(ii) Close to $(\frac{1}{2}, 1)$, we introduce $x = \frac{1}{2} + X(t)$ and $y = 1 + Y(t)$, where $|X| \ll 1$ and $|Y| \ll 1$. The linearised equations are then

$$\dot{X} = 4\left(\frac{1}{2} + X\right) - 4\left(\frac{1}{2} + X\right)(1 + Y) = -2Y + \dots,$$
$$\dot{Y} = -9\,(1 + Y) + 184\left(\frac{1}{2} + X\right)(1 + Y) = 18X + \dots.$$

Seeking a solution of the from $\mathbf{X} = \mathbf{a}\mathrm{e}^{\lambda t}$, we find

$$\begin{vmatrix} -\lambda & -2 \\ 18 & -\lambda \end{vmatrix} = 0,$$

and so $\lambda = \pm 6\mathrm{i}$. Thus the equilibrium point is a 'centre'.
When $\lambda = 6\mathrm{i}$, $\begin{pmatrix} -6\mathrm{i} & -2 \\ 18 & -6\mathrm{i} \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ -3\mathrm{i} \end{pmatrix}$.
When $\lambda = -6\mathrm{i}$, $\begin{pmatrix} 6\mathrm{i} & -2 \\ 18 & 6\mathrm{i} \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ 3\mathrm{i} \end{pmatrix}$.
So the general solution is given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -3\mathrm{i} \end{pmatrix} \mathrm{e}^{6\mathrm{i}t} + c_2 \begin{pmatrix} 1 \\ 3\mathrm{i} \end{pmatrix} \mathrm{e}^{-6\mathrm{i}t} = c_3 \begin{pmatrix} \cos 6t \\ 3\sin 6t \end{pmatrix} + c_4 \begin{pmatrix} \sin 2t \\ -3\sin 6t \end{pmatrix},$$

where $c_1,\ c_2,\ c_3\ \&\ c_4$ are constants.

(b) Close to $(0,0)$ the trajectories are given by

$$\left(\frac{x(t)}{A}\right)^9 = \left(\frac{y(t)}{B}\right)^{-4}$$

and so $[x(t)]^9[y(t)]^4 = \text{constant}$.
Close to $(\frac{1}{2}, 1)$,

$$X^2 + \left(\frac{Y}{3}\right)^2 = c_3^2 + c_4^2$$

So the trajectories are ellipses.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{9y(-1+2x)}{4x(1-y)}$. This is a separable equation and leads to

$$4\left(\log|y| - y\right) = 9\left(2x - \log|x|\right) + \text{constant}.$$

Close to $(0,0)$, $\log|y| \gg |y|$ and $\log|x| \gg |x|$ and so

$$4\log|y| + 9\log|x| = \text{constant},$$

recovering the equation for trajectories derived above.
Close to $(\frac{1}{2}, 1)$, $x = \frac{1}{2} + X$, $y = 1 + Y$. When $|X| \ll 1$,

$$\log|x| = \log|\frac{1}{2}+X| = \log|\frac{1}{2}(1+2X)| = \log\frac{1}{2} + \log|1+2X| = \log\frac{1}{2} + 2X - 2X^2 + \ldots.$$

and when $|Y| \ll 1$,

$$\log|y| = \log|1+Y| = Y - \frac{1}{2}Y^2 + \ldots.$$

Substituting these approximate expressions gives

$$4\left(Y - \frac{1}{2}Y^2 - 1 - Y\right) = 9\left(1 + 2X - \log\frac{1}{2} - 2X + 2X^2\right) + \text{constant}.$$

This simplifies to

$$9X^2 + Y^2 = \text{constant},$$

which is identical to what was derived above.

4. (a) The equilibrium points of this system are determined by simultaneously solving $x(1-x-y) = 0$ and $y(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x) = 0$. This yields four equilibrium points, namely $(x,y) = (0,0)$, $(0,2)$, $(1,0)$ and $(\frac{1}{2}, \frac{1}{2})$.
To determine their nature we must examine the system in the vicinity of each of them.
(i) First close to $(0,0)$, the system becomes

$$\dot{x} = x + \ldots \qquad \text{and} \qquad \dot{y} = \frac{1}{2}y + \ldots,$$

which leads to $x = A\mathrm{e}^t$ and $y = B\mathrm{e}^{t/2}$. Thus all trajectories evolve away from the origin - it is an 'unstable node'.
(ii) Close to $(0,2)$ we write $y = 2 + Y$ and then the linearised equations are given by

$$\dot{x} = x(1-x-2-Y) = -x + \ldots \quad \text{and} \quad \dot{Y} = -(2+Y)\left(\frac{1}{2} - \frac{1}{4}(2+Y) - \frac{3}{4}x\right) = -\frac{3}{2}x - \frac{1}{2}Y$$

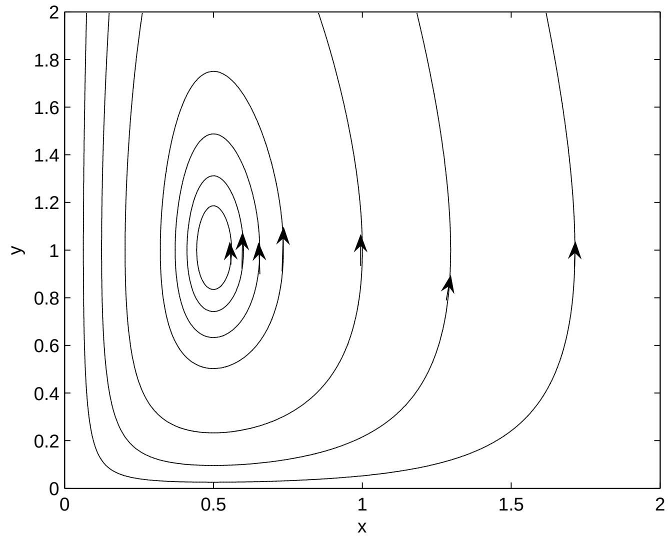

Figure 3: The phase plane for question 3.

Seeking a solution $\mathbf{x} = \mathbf{a}e^{\lambda t}$ demands that

$$\begin{vmatrix} -1-\lambda & 0 \\ -3/2 & -1/2-\lambda \end{vmatrix} = 0$$

and so $\lambda = -1$ and $\lambda = -1/2$. This point is a 'stable node'.
When $\lambda = -1$, $\begin{pmatrix} 0 & 0 \\ -3/2 & 1/2 \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
When $\lambda = -1/2$, $\begin{pmatrix} -1/2 & 0 \\ -3/2 & 0 \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
The general solution close to this equilibrium point is then

$$\begin{pmatrix} x \\ y-2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2}.$$

(iii) Close to $(1,0)$ we write $x = 1 + X$ and then the linearised equations are given by

$$\dot{X} = (1+X)(-X-y) = -X-y+\ldots \quad \text{and} \quad \dot{y} = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}(1+X)\right) = -\frac{1}{4}Y+\ldots$$

Seeking a solution $\mathbf{x} = \mathbf{a}e^{\lambda t}$ demands that

$$\begin{vmatrix} -1-\lambda & -1 \\ 0 & -1/4-\lambda \end{vmatrix} = 0$$

and so $\lambda = -1$ and $\lambda = -1/4$. This point is a 'stable node'.
When $\lambda = -1$, $\begin{pmatrix} 0 & -1 \\ 0 & 3/4 \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
When $\lambda = -1/4$, $\begin{pmatrix} -3/4 & -1 \\ 0 & 0 \end{pmatrix} \mathbf{a} = 0$ and so $\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.
The general solution close to this equilibrium point is then

$$\begin{pmatrix} x-1 \\ y \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_4 \begin{pmatrix} 4 \\ -3 \end{pmatrix} e^{-t/4}.$$

(iv) Close to $(\frac{1}{2}, \frac{1}{2})$ we write $x = \frac{1}{2} + X$, $y = \frac{1}{2} + Y$ and then the linearised equations are given by

$$\dot{X} = (\frac{1}{2} + X)(1 - \frac{1}{2} - X - \frac{1}{2} - Y) = -\frac{1}{2}X - \frac{1}{2}Y + \ldots \quad \text{and}$$
$$\dot{Y} = (\frac{1}{2} + Y)(\frac{1}{2} - \frac{1}{4}(\frac{1}{2} + Y) - \frac{3}{4}(\frac{1}{2} + X)) = -\frac{3}{8}X - \frac{1}{8}Y$$

Seeking a solution $\mathbf{x} = \mathbf{a}e^{\lambda t}$ demands that

$$\begin{vmatrix} -\frac{1}{2} - \lambda & -\frac{1}{2} \\ -\frac{3}{8} & -\frac{1}{8} - \lambda \end{vmatrix} = 0,$$

which simplifies to $\lambda^2 + 5\lambda/8 - 1/8 = 0$. Solving this quadratic gives $\lambda = \lambda_{1,2} = -\frac{5}{16} \pm \frac{\sqrt{57}}{16}$. Since the values for λ are of opposite sign, this point is a 'saddle point'.

When $\lambda = -\frac{5}{16} + \frac{\sqrt{57}}{16}$, $\mathbf{a} = \begin{pmatrix} 1 \\ (-3 - \sqrt{57})/8 \end{pmatrix}$.

When $\lambda = -\frac{5}{16} - \frac{\sqrt{57}}{16}$, $\mathbf{a} = \begin{pmatrix} 1 \\ (-3 + \sqrt{57})/8 \end{pmatrix}$.

The general solution close to this equilibrium point is then

$$\begin{pmatrix} x - 1/2 \\ y - 1/2 \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ (-3 - \sqrt{57})/8 \end{pmatrix} e^{\lambda_1 t} + c_4 \begin{pmatrix} 1 \\ (-3 + \sqrt{57})/8 \end{pmatrix} e^{\lambda_2 t}.$$

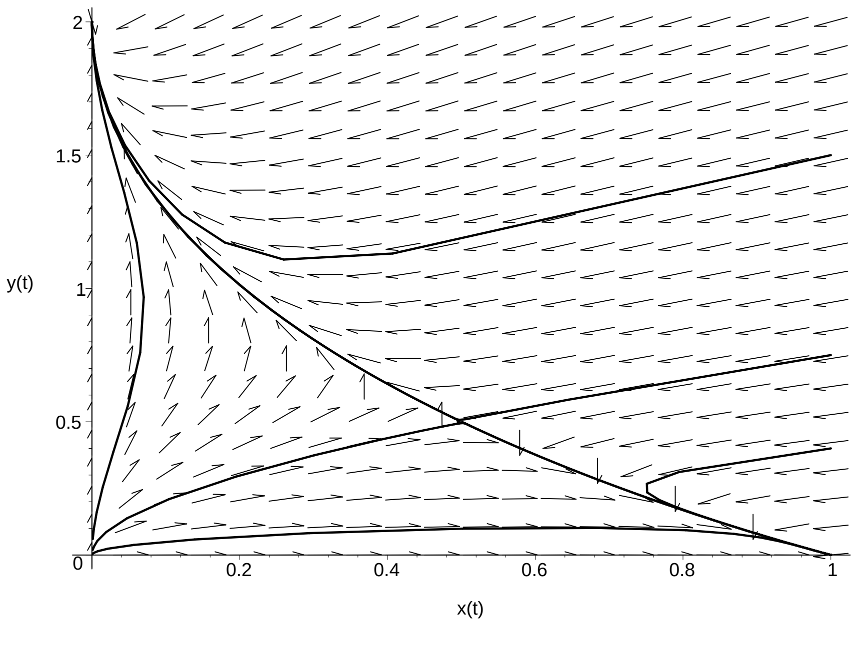

Figure 4: The phase plane for question 4. The solid lines are trajectories; the arrows are the vector field $(\dot{x}, \dot{y})$.

(b) As $t \to \infty$, the system may evolve towards either of the stable nodes, depending upon the starting position. This means that both the possible final states are with one of the species being eliminated (either $x = 0$ or $y = 0$). There is a curve passing through the saddle $(1/, 1/2)$, which separates the phase plane into initial conditions that evolve towards $(0, 2)$ and initial conditions that evolve towards $(1, 0)$. This curve is termed a 'separatrix'.