Calculus (MATH11007): Solutions to Sheet 7 **2012**

First-order differential equations

1. (a) e^x is an integrating factor and so we have $\dfrac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^x) = \mathrm{e}^{2x}$. Hence

$$\mathrm{e}^x y = y(0) + \int_0^x \mathrm{e}^{2t}\,\mathrm{d}t = 1 - \frac{\mathrm{e}^{2x}}{2} - \frac{1}{2} = \frac{1}{2}\left(1+\mathrm{e}^{2x}\right).$$

Thus $y = \cosh x$. This solution is valid for $x \in \mathbb{R}$.

(b) This is separable and we find

$$\int y\,\mathrm{d}y - \int x\,\mathrm{d}x = y^2 - x^2 = C\,.$$

Hence $y = \sqrt{C+x^2}$.

(c) x^2 is an integrating factor and we have $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x^2 y\right] = x^4$. Hence

$$x^2 y(x) = \int x^4 \mathrm{d}x + C = \frac{x^5}{5} + C$$

and so $y = x^3/5 + C/x^2$.

(d) This is separable.

$$\int_1^y \frac{\mathrm{d}u}{u^2} = \int_0^x 2t\,\mathrm{d}t$$

and so

$$\frac{-1}{y(x)} + 1 = x^2\,, \quad \text{i.e. } y(x) = \frac{1}{1-x^2}\,.$$

The range of validity is $|x| < 1$.

(e) $\sqrt{1+t^2}$ is an integrating factor. So $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\sqrt{1+t^2}\,x\right] = t\sqrt{1+t^2}$ Hence

$$\sqrt{1+t^2}\,x(t) - 1 = \int_0^t \tau\sqrt{1+\tau^2}\,\mathrm{d}\tau \overset{u=1+\tau^2}{\underset{\downarrow}{=}} \frac{1}{2}\int_1^{1+t^2} \sqrt{u}\,\mathrm{d}u = \frac{1}{3}\left[(1+t^2)^{\frac{3}{2}} - 1\right].$$

Therefore

$$x(t) = \frac{1}{3}\left[1 + t^2 + \frac{2}{\sqrt{1+t^2}}\right].$$

This is valid for $t \in \mathbb{R}$.

(f) This is separable. Hence

$$\int_2^s \frac{\mathrm{d}u}{u^2} = \int_0^t \sin\tau\,\mathrm{d}\tau\,, \quad \text{i.e. } s(t) = \frac{1}{\cos t - \frac{1}{2}}\,.$$

This is valid for $|t| < \pi/3$.

(g) This is separable.

$$\int \frac{\mathrm{d}p}{p} = \int \frac{s}{s^2+1}\,\mathrm{d}s = \ln\sqrt{1+s^2} + \ln C\,.$$

Hence $p = C\sqrt{1+s^2}$.

(h) $\mathrm{e}^{\sin t}$ is an integrating factor. Hence $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\sin t}u\right] = \cos t\,\mathrm{e}^{\sin t}$. So

$$u = \mathrm{e}^{-\sin t}\left[\int^t \mathrm{e}^{\sin z}\cos z\,\mathrm{d}z + C\right] = \mathrm{e}^{-\sin t}\left[\mathrm{e}^{\sin t} + C\right] = 1 + C\mathrm{e}^{-\sin t}\,.$$

(i) e^{x^2} is an integrating factor. So $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x^2}y\right] = 2x^3\mathrm{e}^{x^2}$. Hence

$$y(x) = \mathrm{e}^{-x^2}\left[\int^x 2t^3\mathrm{e}^{t^2}\,\mathrm{d}t\right] = \mathrm{e}^{-x^2}\left[x^2\mathrm{e}^{x^2} - \int^x 2t\mathrm{e}^{t^2}\,\mathrm{d}t\right]$$
$$= \mathrm{e}^{-x^2}\left[(x^2-1)\mathrm{e}^{x^2} + C\right] = x^2 - 1 + C\mathrm{e}^{-x^2}\,.$$

(j) We write $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{-1}{2+\frac{x}{t}}$ and therefore recognise that the equation is homogeneous. With the substitution $u = x/t$, the equation is transformed to

$$t\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{-1}{2+u} - u = -\frac{(1+u)^2}{2+u}\,.$$

This is separable:

$$\int \frac{\mathrm{d}t}{t} = -\int \frac{2+u}{(1+u)^2}\mathrm{d}u = -\int \left[\frac{1}{1+u} + \frac{1}{(1+u)^2}\right]\mathrm{d}u = -\ln|1+u| + \frac{1}{1+u} + \ln C\,.$$

and so we find

$$|t| = \frac{C}{|1+x/t|}\mathrm{e}^{\frac{1}{1+x/t}}\,.$$

2. (a) The equation is separable:

$$\int \frac{\mathrm{d}y}{\sqrt{4-y^2}} = \int \frac{\mathrm{d}x}{1-x^2} \qquad \text{leads to} \qquad y = 2\sin\left(\ln\sqrt{\left|\frac{x+1}{x-1}\right|} + C\right)\,.$$

(b) e^{ax} is an integrating factor and so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{ax}y\right] = b\mathrm{e}^{ax}\,.$$

Hence

$$y(x) = \mathrm{e}^{-ax}\left[b\int^x \mathrm{e}^{at}\,\mathrm{d}t + C\right] = \begin{cases} bx + C & \text{if } a = 0 \\ \frac{b}{a} + C\mathrm{e}^{-ax} & \text{otherwise} \end{cases}\,.$$

(c) Factorising gives $(y'-2)(y'-1) = 0$. So there are two solutions: $y(x) = 2x$ and $y(x) = x+1$.

3. On substituitng $y(x) = u(x)^\alpha$, we find $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \alpha u^{\alpha-1}\dfrac{\mathrm{d}u}{\mathrm{d}x}$. This transforms the original equation to

$$\alpha\frac{\mathrm{d}u}{\mathrm{d}x} = fu + gu^{\nu\alpha-\alpha+1}.$$

In this expression we are free to select the value of α. A convenient choice is such that $(\nu-1)\alpha + 1 = 0$, which implies that $\alpha = 1/(1-\nu)$. Then the equation ahs become

$$\frac{1}{1-\nu}\frac{\mathrm{d}u}{\mathrm{d}x} = fu + g.$$

(a) Choose $\alpha = -1/2$, so $y(x) = u(x)^{-1/2}$ and the problem becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 2xu = -2x, \qquad u(0) = 2.$$

Integrating factor is e^{-x^2} and the solution is given by

$$\frac{1}{y(x)^2} = u(x) = 1 + \mathrm{e}^{x^2}.$$

(b) Choose $\alpha = 1/2$, so $\sigma(t) = u(t)^{1/2}$ and the problem becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} - 2u = 2t, \qquad u(0) = 1.$$

Integrating factor is e^{-2t} and the solution is given by

$$\sigma(t)^2 = u(t) = -t - \frac{1}{2} + \frac{3}{2}\mathrm{e}^{2t}.$$

4. This is separable. $y = 1$ and $y = -1$ are solutions. There are other solutions, valid in certain regions. We consider two cases:

(a) $|x| < 1$. Then

$$\frac{\mathrm{d}y_\pm}{\sqrt{1-y_\pm^2}} = \frac{\pm\mathrm{d}x}{\sqrt{1-x^2}}$$

and the solution is $y_\pm(x) = \sin(\pm\arcsin x + C)$.

(b) $|x| > 1$. Then

$$\frac{\mathrm{d}y_\pm}{\sqrt{y_\pm^2-1}} = \frac{\pm\mathrm{d}x}{\sqrt{x^2-1}}$$

and the solution is $y_\pm(x) = \cosh(\pm\mathrm{arccosh}x + C)$.

5. (a) This is separable:

$$\int_0^{u(t)} \frac{\rho_s d^2}{\Delta\rho g d^2 - 18\mu s}\,\mathrm{d}s = \int_0^t \mathrm{d}v.$$

On integrating and re-arranging, we find that

$$u(t) = \frac{\Delta\rho g d^2}{18\mu}\left(1 - \exp(-\frac{18\mu t}{\rho_s d^2})\right).$$

(b) The terminal velocity $V_s = \lim_{t \to \infty} u(t) = \dfrac{\Delta\rho g d^2}{18\mu}$.

(c) Solving for $u(t) = V_s(1 - \mathrm{e}^{-1})$, we determine $t = t_s = \rho_s d^2/[18\mu]$.

(d) For 100μm sand particles in water, $V_s = 0.87\mathrm{cm\,s^{-1}}$ and $t_s = 0.0014$s.

6. This equation is separable

$$\int_0^{x(t)} \frac{1}{(a-s)(b-s)}\,\mathrm{d}s = K\int_0^t \mathrm{d}v.$$

Hence $\dfrac{b-x}{a-x} = \dfrac{b}{a}\mathrm{e}^{K(b-a)t}$. and so

$$x(t) = \frac{ab(1-\mathrm{e}^{K(b-a)t})}{a - b\mathrm{e}^{K(b-a)t}}.$$

Hence as $t \to \infty$ $x(t) \to b$ if $b < a$ and $x(t) \to a$ if $a < b$.