Calculus (MATH11007): Solutions 8 **2012**

First-order differential equations

1. The differential equation is separable. However if $P(0) = \hat{P} = \beta/\delta$ then $\mathrm{d}P/\mathrm{d}t = 0$ and $P(t) = \hat{P}$ prvoides the solution for (ii). For parts (i) and (iii)

$$\int_{P_0}^{P(t)} \frac{1}{P'(\beta - \delta P')}\,\mathrm{d}P' = \int_0^t \mathrm{d}s, \text{ which gives } \ln\left|\frac{P(\hat{P}-P_0)}{P_0(\hat{P}-P)}\right| = \beta t.$$

Thus in both (i) and (iii), we deduce that

$$P(t) = \frac{\hat{P}P_0}{(\hat{P}-P_0)\mathrm{e}^{-\beta t} + P_0}.$$

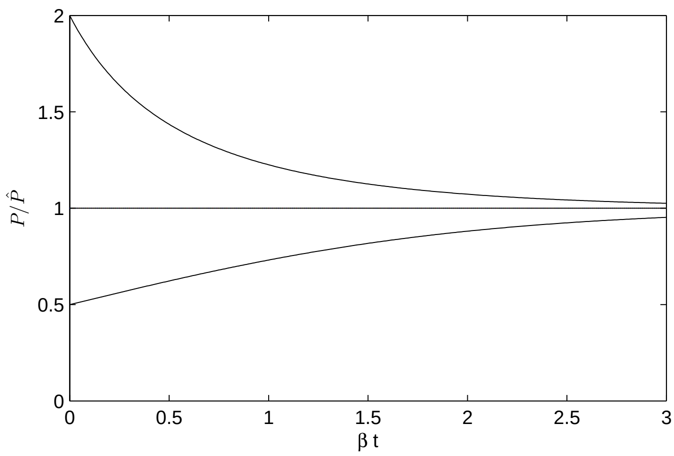

Figure 1: The population, $P(t)/\hat{P}$, as a function of βt for (i) $P_0/\hat{P} = 2$; (ii) $P_0/\hat{P} = 1$; and (iii) $P_0/\hat{P} = 0.5$.

2. First write the differential equation as $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{y^2+x}{y}$. This is then in the form for which an integrating factor may be introduced

$$\frac{\mathrm{d}x}{\mathrm{d}y} - \frac{x}{y} = y \text{ which gives } x = y^2 + Cy.$$

3. From direct integration we find $y = \displaystyle\int_0^x \exp(\sin(\mathrm{s}))\ \mathrm{ds}$. But $-1 \le \sin(s) \le 1$ and so

$$x\mathrm{e}^{-1} \le y(x) \le x\mathrm{e}.$$

4. Both parts have separable differential equations

(a) $\displaystyle\int_1^{x(t)} \frac{1}{x'^3}\,\mathrm{d}x' = -\int_0^t \frac{1}{2}\,\mathrm{d}s$, which gives $x(t) = (t+1)^{-1/2}$, valid for $t > -1$.

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(b) $\displaystyle\int_1^{x(t)} \frac{1}{x'^3}\,\mathrm{d}x' = \int_0^t \frac{1}{2}\,\mathrm{d}s$, which gives $x(t) = (1-t)^{-1/2}$, valid for $t < 1$.

5. (a) $x(t) = \left(\dfrac{t+2}{2}\right)^2.$

   (b) $x(t) = \left(\dfrac{t}{2}\right)^2.$

   (c) $x(t) = 0.$

   The function $\sqrt{x}$ is not differentiable at $x = 0$ and so a unique solution is not guaranteed.

6. Separating variables $\displaystyle\int \mathrm{d}x = \int \frac{1}{(2+y)(1-y)}\,\mathrm{d}y = \int \frac{1}{3}\left(\frac{1}{2+y} + \frac{1}{1-y}\right)\mathrm{d}y$. Hence we deduce

$$\ln\left|\frac{2+y}{1-y}\right| = 3x + C.$$

   (a) The condition $y(0) = 0$ implies $\left|\dfrac{2+y}{1-y}\right| = 2\mathrm{e}^{3x}$. Thus the solution is

$$y = \frac{2\mathrm{e}^{3x} - 2}{1 + 2\mathrm{e}^{3x}},$$

   because $1 > y(x)$. This solution is valid for all values of x.

   (b) The condition $y(0) = 4$ implies that $\left|\dfrac{2+y}{1-y}\right| = 2\mathrm{e}^{3x}$. But now the solutions is

$$y = \frac{2\mathrm{e}^{3x} + 2}{2\mathrm{e}^{3x} - 1},$$

   because $1 < y(x)$. This solution is valid for $x > -\frac{1}{3}\ln 2$.

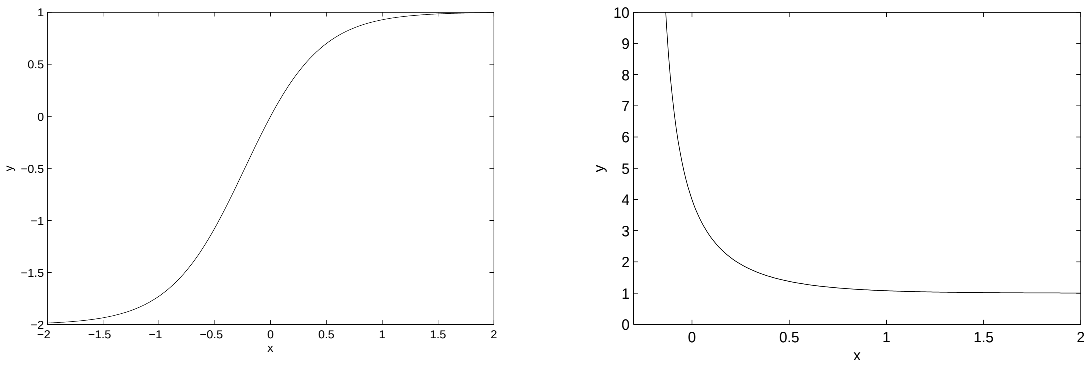

Figure 2: The solutions to question 6(i) and (ii)

7. (a) The differential equation has solution

$$\phi(x) = \sin x + \frac{\cos x + C}{x},$$

   where C is a constant of integration.

   (i) When $\phi(0) = 0$, $C = -1$ and $\phi(x) = \sin x + \dfrac{\cos x - 1}{x}$.

   (ii) When $\phi(0) = 1$, no value of C can be found to satisfy the boundary condition.

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(b) Writing $\frac{\mathrm{d}\phi}{\mathrm{d}x} = f(\phi, x) = \frac{x\cos x - \phi}{x}$, we see that: (i) $f(0,0)$ is defined; but (ii) $f(1,0)$ is singular. Thus in (ii) the existence of a solution is not guaranteed.

8. (a) The solution is $y(x) = \mathrm{e}^x$ and so $y(a) = \mathrm{e}^a$.

(b) Euler's method gives $y_{n+1} = y_n + hf(nh, y_n)$. Thus $y_{n+1} = (1+h)y_n$. Hence given $y_0 = 1$, $y_n = (1+h)^n$.

(c) Suppose there are N equally spaced steps so that $h = a/N$. Then

$$y_N = \left(1 + \frac{a}{N}\right)^N \to e^a \text{ as } N \to \infty.$$

9. (a) The exact solution is $y(t) = 6 - 2\mathrm{e}^{-t} - 3\mathrm{e}^{-t/2}$.

(b) The figure shows the exact curve with the approximations with (i) $h = 1$; (ii) $h = 0.1$; and (iii) $h = 0.01$. For the latter case, it is difficult to distinguish it from the exact result.

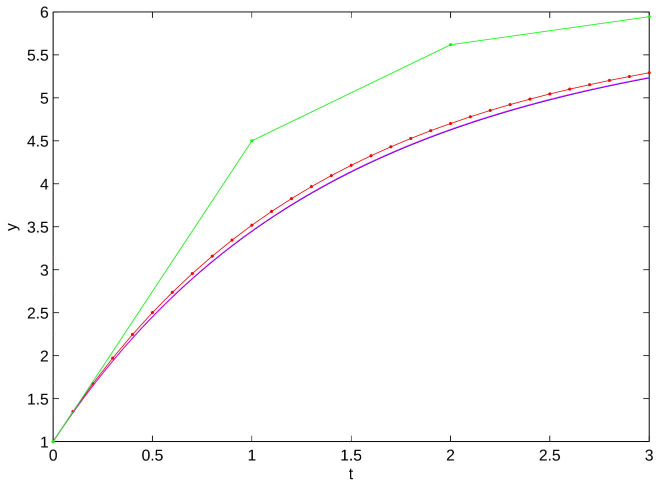

Figure 3: The exact solution to question 9, together with the numerical approximations using Euler's method with $h = 1$, 0.1 and 0.01.

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