Calculus (MATH11007): Solutions 9 — 2012

Second-order differential equations

1. (a) Seeking a solution of the form $y(x) = \mathrm{e}^{rx}$ we find that

$$r^2 + 5r + 6 = (r+3)(r+2) = 0.$$

Thus the general solution is $y(x) = A\mathrm{e}^{-3x} + B\mathrm{e}^{-2x}$, where A and B are constants.

(b) Applying the condition $y(0) = 1$ implies $A + B = 1$.
Applying the condition $y'(0) = 0$ implies $-3A - 2B = 0$. Thus $A = -2$, $B = 3$ and the solution is

$$y(x) = -2\mathrm{e}^{-3x} + 3\mathrm{e}^{-2x}.$$

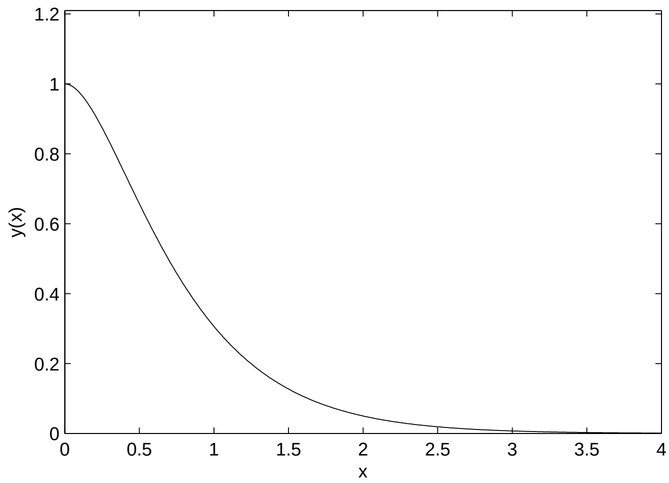

Figure 1: The solution $y(x) = -2\mathrm{e}^{-3x} + 3\mathrm{e}^{-2x}$ as a function of x (Question 1).

2. Seeking a solution of the form $x(t) = \mathrm{e}^{rt}$ we find that

$$r^2 - 4r + 1 = (r-2)^2 - 3 = 0.$$

Thus $r = 2 \pm \sqrt{3}$ and the general solution is $x(t) = A\mathrm{e}^{(2+\sqrt{3})t} + B\mathrm{e}^{(2-\sqrt{3})t}$. Applying $x(0) = 0$, we deduce $A + B = 0$ and from $\mathrm{d}x/\mathrm{d}t(0) = 1$, $(2+\sqrt{3})A + (2-\sqrt{3})B = 1$. Thus $A = -B = 1/[2\sqrt{3}]$ and the solution is

$$x(t) = \frac{1}{2\sqrt{3}}\left(\mathrm{e}^{(2+\sqrt{3})t} - \mathrm{e}^{(2-\sqrt{3})t}\right) = \frac{\mathrm{e}^{2t}}{\sqrt{3}}\sinh\left(\sqrt{3}t\right).$$

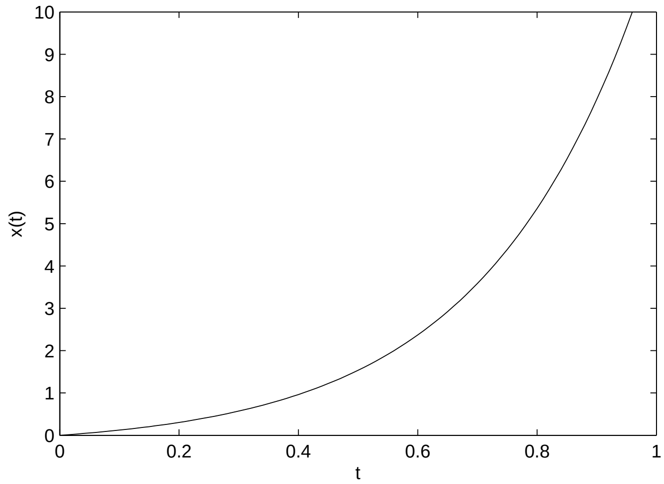

Figure 2: The solution $x(t) = \frac{e^{2t}}{\sqrt{3}} \sinh\left(\sqrt{3}t\right)$ as a function of t (Question 2).

3. Seeking a solution of the form $y(x) = e^{rx}$ we find that

$$r^2 + 9 = (r - 3i)(r + 3i) = 0.$$

Thus $r = \pm 3i$ and the general solution is $y(x) = A \sin 3x + B \cos 3x$.

   (a) When $y(0) = 0$ and $y'(0) = 1$, we find that $y(x) = \frac{1}{3} \sin 3x$.
   (b) When $y(0) = 1$ and $y'(0) = 0$, we find that $y(x) = \cos 3x$.

4. Seeking a solution of the form $y(x) = e^{rx}$ to the homogeneous problem, we find that

$$r^2 + 4r + 3 = (r + 1)(r + 3) = 0.$$

Thus $r = -1, -3$ and the complementary function is $y(x) = Ae^{-x} + Be^{-3x}$.

   (a) Try $y(x) = Ce^{2x}$ and so $4C + 8C + 3C = 1$, thus the general solution is

$$y(x) = \frac{1}{15}e^{2x} + Ae^{-x} + Be^{-3x}.$$

   (b) Try $y(x) = ax^2 + bx + c$ and so $2a + 4(2ax + b) + 3(ax^2 + bx + c) = x^2$. Thus the general solution is

$$y(x) = \frac{x^2}{3} - \frac{8x}{9} + \frac{26}{27} + Ae^{-x} + Be^{-3x}.$$

   (c) Try $y(x) = C \sin 5x + D \cos 5x$ and so $(-25C - 20D + 3C) \sin 5x + (-25D + 20C + 3D) \cos 5x = \sin 5x$. Thus the general solution is

$$y(x) = -\frac{5}{221} \cos 5x - \frac{11}{442} \sin 5x + Ae^{-x} + Be^{-3x}.$$

   (d) e^{-x} is in the complementary function, so try $y(x) = Cxe^{-x}$. Thus $(-2C + 4C)e^{-x} = e^{-x}$ and the general solution is

$$y(x) = \frac{xe^{-x}}{2} + Ae^{-x} + Be^{-3x}.$$

5. Seeking a solution of the form $y(x) = \mathrm{e}^{rx}$, we find that

$$r^2 - 4r + 8 = (r-2)^2 + 4 = 0.$$

Thus $r = 2 \pm 2\mathrm{i}$ and so the general solution is $y(x) = \mathrm{e}^{2x}(A \sin 2x + B \cos 2x)$. Then enforcing $u(0) = 0$ implies $B = 0$.

   (a) If $u(\pi/2) = 0$, then A remains undetermined.

   (b) If $u(1) = 0$, then $A = 0$ and the only solution is $u(x) = 0$.

6. Seeking a solution to the homogeneous problem of the form $z(x) = \mathrm{e}^{rx}$, we find that

$$r^2 - 6r + 9 = (r-3)^2 = 0.$$

So there is a repeated root and the complementary function is $z(x) = (A + Bx)\mathrm{e}^{3x}$. To find the particular integral we try $z(x) = Cx^2\mathrm{e}^{3x}$. Then $2C\mathrm{e}^{3x} = \mathrm{e}^{3x}$ and so the general solution is

$$z(z) = (A + Bx)\mathrm{e}^{3x} + \frac{x^2\mathrm{e}^{3x}}{2}.$$

7. From Q3, the solution satisfying the differential equation and $y(0) = 0$ is $y(x) = A \sin 3x$. If $y(L) = 0$ then $\sin 3L = 0$ and so $3L = m\pi$ with m an integer.

8. Seeking a solution of the form $y(x) = \mathrm{e}^{rx}$, we find that

$$r^2 + 2r + 1 + \omega^2 = (r+1)^2 + \omega^2 = 0.$$

So $r = -1 \pm \mathrm{i}\omega$ and the general solution is $y(x) = \mathrm{e}^{-x}(A \sin \omega x + B \cos \omega x)$. Enforcing $y(0)$ gives $B = 0$ and enforcing $y(\pi) = 0$ implies that $\omega = m$ (with m an integer) for a non-trivial solution.

9. (a) Seeking a solution of the form $y(x) = \mathrm{e}^{rx}$, we find that

$$r^2 - 2r + 1 = (r-1)^2 = 0.$$

   Thus $r = 1$ is a repeated root and $y(x) = (Ax + B)\mathrm{e}^x$. After applying boundary conditions we find that $y(x) = x\mathrm{e}^x$.

   (b) Seeking a solution of the form $y(x) = \mathrm{e}^{rx}$, we find that

$$r^2 - (2+\epsilon)r + 1 + \epsilon = (r-1)(r-1-\epsilon) = 0.$$

   Thus the general solution is $y(x) = A\mathrm{e}^x + B\mathrm{e}^{(1+\epsilon)x}$. After applying boundary conditions we find that $y(x) = -\frac{\mathrm{e}^x}{\epsilon} + \frac{\mathrm{e}^{(1+\epsilon)x}}{\epsilon}$.
   Now consider the limit $\epsilon \to 0$ and note that $(e^{\epsilon x} - 1)/\epsilon \to x$, so that $y(x) \to x\mathrm{e}^x$ (recovering solution to (a)).