Length and Time Scales of Response of Sediment Suspensions to Changing Flow Conditions

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Abstract: Turbulent suspensions of sediment are investigated to establish the characteristic length and time scales on which they adjust from one state to another. The suspensions are modeled by using a simple closure for the turbulent fluctuations in which the average flux of sediment is treated as a diffusion process. A key dimensionless settling parameter, which is closely related to the Rouse number, measures the magnitude of the settling to diffusive fluxes of particles. It is shown how the length and time scales on which the suspension responds are a function of the settling parameter and the assumed form of the eddy diffusivity, and that the predictions are broadly in accord with laboratory experiments. It is further established analytically that, in the regimes of the settling parameter much greater or much less than unity, the timescale of response is independent of the form of the eddy diffusivity. This motivates the use of simple eddy diffusivity laws to provide generic insight to the unsteady evolution of complex suspension and sedimentation problems. **DOI: 10.1061/(ASCE)HY.1943-7900.0000532.**

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Introduction

Accurate modeling of suspensions of sedimentary particles in a turbulent flow is an important challenge in coastal and hydraulic engineering. In particular, quantitative predicting the rate at which sediment is eroded or deposited is key to assessing morphological change resulting from variations in environmental conditions. In this paper, the authors analyze the response of suspensions resulting from spatial and temporal changes in the suspending flow and identify how the length and time scales on which the suspension responds depend on the settling velocity of the suspended particles, the mean velocity of the suspending fluid, the depth of the flow and the intensity of the turbulence (as measured through the magnitude of the turbulent friction velocity).

Mathematically modeling dilute turbulent suspensions of noncohesive particles poses many challenges because of the absence of complete models that capture fully the complicated interactions between the fluid and the particles (Dyer and Soulsby 1988; Fredsoe and Deigaard 1992). However, a common approach is to assume that the turbulence-induced flux of particulate may be expressed as a diffusive flux, with the sediment diffusivity (more commonly termed the eddy diffusivity) determined empirically (Dyer and Soulsby 1988). Conservation of particulate mass leads to an advection-diffusion-settling equation in which the unsteady variation of the concentration of particles is balanced by advection, with the mean fluid flow, settling, and diffusion resulting from the effects of the turbulence. This has led to predictions of fully developed profiles of sediment concentration that compare reasonably well with experimental measurements, when coupled with appropriate empirical expressions for the magnitude of the turbulence effects (Rouse 1938; van Rijn 1984b; Dyer and Soulsby 1988).

Difficulties inevitably exist with this approach to modeling turbulent suspensions. It depends crucially on the specification of the eddy diffusivity, which is often related to the eddy viscosity, which in turn prescribes the rate at which the fluid momentum is mixed. The theoretically predicted, steady-state profiles of sediment depend strongly on the boundary conditions imposed at the bed. However, these bed conditions are also not firmly established (Dyer and Soulsby 1988). In addition, it is difficult to account for particle interactions, which play a role in nondilute suspensions. [The recent approaches by Drew (1983), Jenkins and Hanes (1998), and Hsu et al. (2004) address the latter by utilizing Favre averaging of the turbulent fluctuations and by introducing an explicit expression of momentum balance for the suspended particles.] Despite these difficulties, a diffusion-based model of turbulent suspensions is widely employed (Soulsby 1998) and provides insight into the evolution of suspensions. This paper will pursue this approach.

This paper analyzes the response of a suspension of sediment in a turbulent flow to a change in the flow speed, and consequently a change in the ability of the turbulent fluctuations to support the relatively dense particulate material (or to a change in the sediment supply) so that the suspension is no longer in equilibrium. These types of flow changes can occur in a number of physical settings. For example, tidal flows vary temporally and spatially, and both Prandle (1997) and Pritchard (2006) analyzed the temporal response of suspensions in this setting. Large-scale particulate flows such as dilute turbidity currents progressively decelerate and lose their particle-bearing capability. This leads to sedimentation of particulate material and the formation of a deposit along the underlying boundary. Motivated by this application, Sumner et al. (2008) conducted experiments to study the rate of growth and composition of the deposit from a polydisperse suspensions within an annular flume. By decreasing the rate of its rotation, they generated

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suspensions that were dependent only upon time and the vertical coordinate. Suspensions within steady but spatially evolving flows have also been studied in the laboratory. Depositional flows were created by introducing a sediment excess at a location. This is subsequently deposited downstream as the suspension adjusts to a new equilibrium (Apmann and Rumer 1970; Jobson and Sayre 1970b; Ashida and Okabe 1982; Celic and Rodi 1988). However, nonuniform erosional flows have been created by studying flow and sediment transport when the basal boundary abruptly changes from nonerosional to erosional (van Rijn 1984a; Ashida and Okabe 1982).

The analysis in this paper of the response of suspensions from one state to another builds on some previous studies. By using a model in which the eddy diffusivity is constant, Tu et al. (1993) and Pritchard (2006) analytically calculated the temporal response of a suspension, whereas Stansby and Awang (1998) numerically determined timescales when the eddy diffusivity varies quadratically with distance from the boundary. For a steady but spatially evolving flow that encounters an abrupt change in sediment supply, Mei (1969) and Hjelmfelt and Lenau (1970) calculated the response by using a diffusion model of the sediment suspension, whereas Celic and Rodi (1988) tackled the problem numerically by using a more complete model of flow and sediment dynamics. Finally, Claudin et al. (2011) have results that encompass temporal and spatial adjustments, recovering many of the previous results, and illustrate that they and reproduce some previous experimental observations.

The calculations in this paper are similar in approach to the aforementioned new studies, differing somewhat in the choice of the boundary condition. (In this paper, a flux boundary condition is applied at the erodible bed, rather than a reference concentration, which will be subsequently explained.) What is crucial is that it is shown analytically that the length and time scales of response do not depend strongly on the precise form of the eddy diffusivity and are independent of this form in the regimes when the settling velocity of the particles is much greater than or much less than the bed friction velocity. This relative invariance of these scales to the precise empirical representation of the eddy diffusivity is an important result and permits rather general conclusions to be drawn about the unsteady and inhomogeneous behavior of the suspension. The relative invariance to the form of the eddy diffusivity is reminiscent of the results of Dyer and Soulsby (1988) in which they calculated the distribution of steady suspensions and the associated particulate fluxes for a variety of assumed forms of the diffusivity. They showed that for these steady states, the functional form plays only a relatively weak role in determining the overall characteristics of the suspension. In this paper, a somewhat analogous result is established for the unsteady and inhomogeneous response of a suspension.

The paper is structured as follows. A model of a turbulent dilute suspension is first introduced, along with the relevant boundary conditions. Also, the key dimensionless parameter, the Rouse number, is identified, which measures the settling velocity relative to the turbulent bed friction velocity. Then the response of the suspension to an abrupt change in the bed friction velocity or to the supply of sediment is analyzed. The suspension tends toward a new steady state and the difference between the initial and final states decays exponentially in time or space. This is followed by calculating the length and time scale of response as the longest characteristic scale in the exponential decay. The scales of response for varying forms of eddy diffusivity are evaluated as a function of a settling parameter, which is closely related to the Rouse number. It is demonstrated that the scales of response are independent of the functional form of the diffusivity to the leading order when the regimes of the settling parameter are much greater or much less than unity. The theoretical findings for the longest length and time scales on which the suspension responds are compared with data drawn from a series of previously published laboratory experiments. Finally, the results are summarized and some conclusions are drawn.

Modeling Dilute Suspensions in Turbulent Flows

The average concentration, $\phi(x,z,t)$, of a dilute monodisperse suspension of particles within a turbulent flow with mean flow field of $\mathbf{u}=(u(z),0)$ can be modeled by using an expression of mass conservation, which is given by

$$\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}-w_s\frac{\partial\phi}{\partial z}=\frac{\partial}{\partial z}\left(K_z\frac{\partial\phi}{\partial z}\right)+\frac{\partial}{\partial x}\left(K_x\frac{\partial\phi}{\partial x}\right) \quad (1)$$

This equation is applied between the erodible bed, located at $z=0$, and the free surface at $z=h$. In this expression, w_s denotes the settling velocity of the sediment and K_z and K_x denote the vertical and horizontal eddy diffusivities, respectively. Transport associated with vertical diffusion $[(\frac{\partial}{\partial z}(K_z\frac{\partial\phi}{\partial z}))]$ is often assumed to far exceed the streamwise diffusive flux $[(\frac{\partial}{\partial x}(K_x\frac{\partial\phi}{\partial x}))]$, and therefore the latter is often neglected (Mei 1969; Jobson and Sayre 1970a; Stansby and Awang 1998). In this paper, the concentration field has been averaged over the eddy turnover time and is assumed to evolve on slower timescales. The diffusive flux, $-K_z\partial\phi/\partial z$, represents the vertical flux of sediment resulting from turbulent fluctuations. It is equal to the average of the product of the fluctuations of the vertical velocity and the concentrations fields. Representing this transport as a diffusion process is rather crude and fails to capture some features of the dynamics of the suspension; however, it is a commonly used empirical closure of turbulent models and may provide useful quantitative predictions in many situations (Soulsby 1998). Molecular diffusive effects are neglected, which for sand-sized particles in water are certainly negligible and the settling velocity is determined by balancing the submerged weight with the drag. If the sediment is spherical and sufficiently small (formally measured by the Reynolds number on the basis of the settling velocity and particle size as a much smaller unity), then this balance yields Stokes's settling velocity, $w_s=g(\rho_s-\rho_f)d^2/[18\mu]$, where g denotes gravitational acceleration; d, the diameter of the particle; ρ_s and ρ_f, the densities of the sediment and suspending fluid, respectively; and μ, the dynamic viscosity of the fluid. If the particle Reynolds number $(\rho_f w_s d/\mu)$ is not small, then the settling velocity more generally may be calculated by using empirical drag laws (Soulsby 1998).

It is possible to write the eddy diffusivity (Soulsby 1998) as

$$K_z=\kappa u_*(t)Lf\left(\frac{z}{L}\right) \quad (2)$$

where $\kappa=0.4$ is von Kármán's constant; $u_*(t)$ denotes the time-dependent bed friction velocity, as related to the bed shear stress; $\tau_0=\rho_f u_*^2$; L is an appropriate lengthscale for the flow, which is often chosen to be h; and $f(z/h)$ is a shape function that attains a maximum value of unity. For simplicity only planar beds are treated so that no form drag exists and the basal shear stress is equal to the skin friction (Fredsoe and Deigaard 1992).

It is further assumed that the particle concentration is sufficiently dilute to have a negligible effect on flow velocity. For steady flows, the shear stress must vary linearly between the bed and free surface and is given by $\tau=\tau_0(1-z/h)$. The velocity field is then found by using the eddy viscosity formulation $\tau = \rho\nu_e \partial u/\partial z$, where ν_e is the empirical eddy viscosity. The eddy diffusivity, K_z, is often assumed to be equal to the eddy viscosity ν_e; however, the authors note that several studies have investigated situations when their ratio is not unity and may vary systematically with properties of the suspension (Dyer and Soulsby 1988). For the purpose of this investigation and to yield the results on the invariance of the length and time scales to function form of eddy diffusivity, it is not necessary to invoke this assumption.

Boundary Conditions

Because the flow is dilute, the depth of any material deposited from the flow is assumed negligibly small in comparison with the flow depth, which remains constant. This implies that any material deposited from suspension or eroded from the bed is "lost" or "gained," respectively, from the model. The advection-diffusion-settling in Eq. (1) is solved by using two boundary conditions that are applied at the bed ($z = 0$) and at the free surface ($z = h$), and a condition for the initial suspension of particles.

The boundary condition at the free surface is given by a zero flux condition

$$K_z \frac{\partial \phi}{\partial z} + w_s \phi = 0 \quad \text{at} \quad z = h \tag{3}$$

At the base is the potential for mass exchange because of erosion and deposition. Thus, the authors impose

$$-K_z \frac{\partial \phi}{\partial z} = q(t) \qquad \text{at} \qquad z = 0 \tag{4}$$

where $q(t)$ is the basal flux function. In this paper, a popular empirical closure is used for $q(t)$, which expresses the erosion rate of the flow (Dyer and Soulsby 1988; Pritchard and Hogg 2002) in the excess Shields parameter, $(\theta - \theta_c)$, where $\theta = \rho u_*^2/[(\rho_s - \rho)gd]$ and θ_c is the critical Shields parameter for incipient motion. This erosion rate models the vertical flux of particulate material at the flow bed, and is given by

$$q(t) = \begin{cases} m_e\left(\frac{\theta(t)}{\theta_c} - 1\right)^p & \text{for}\, \theta(t) \geq \theta_c \\ 0 & \text{for}\, \theta(t) < \theta_c \end{cases}; \tag{5}$$

where m_e is a dimensional constant, specifying the capacity of the flow (van Rijn 1984a). The exponent p is a dimensionless parameter that is typically in the range of $p = 1$ to 3.5 (Pritchard and Hogg 2002).

The suspensions analyzed in this paper are potentially unsteady and the changes were investigated to the concentration of sediment as the conditions of the flow were varied. This suggests that it is inappropriate to specify a reference concentration at the base of the flow ($z = 0$) in which the concentration is determined as a function of the instantaneous Shields parameter. This is easily justified by considering the response of the suspension to an instantaneous change in the ability to suspend material. For example, a reference concentration relationship would instantaneously adjust to a new concentration value at the bed. In the interior above the boundary, however, the concentration can only adjust progressively through settling and diffusive processes. If the ability to support sediment were abruptly decreased, then the imposition of the reference concentration might indeed introduce a gravitationally unstable vertical profile of concentration. The basal concentration is instead better specified although a flux condition in which the vertical flux of particulate is determined by a function that reflects the conditions at the bed and by the ability of the flow to erode particles [see Cao and Carling (2002) and references therein].

Dimensional Analysis

Vertical length and time scales of the flow are rendered dimensionless by the flow depth and the terminal settling velocity. In this way their dimensionless counterparts are given by $T = w_s t/h$ and $Z = z/h$. The dimensionless downstream flow velocity, U, is expressed as the ratio of the flow velocity to u_*/κ, so that $U = \kappa u/u_*$. This choice anticipates that the flow profile is logarithmic [$U = \log(1 + Z/Z_0)$]. Furthermore, it is convenient to define the dimensionless streamwise coordinate, $X = \kappa w_s x/(h u_*)$, because this choice ensures that the unsteady and streamwise advective terms in Eq. (1) are free of dimensionless ratios. The important dimensionless parameter in the problem is the settling parameter β, which measures the relative magnitude of the advective (i.e., settling) to diffusive processes. The settling parameter is given by

$$\beta = \frac{w_s h}{K_m} \tag{6}$$

where K_m is the maximum magnitude of the eddy diffusivity $[K_z(z,t)]$. It is time dependent; later, flows will be analyzed in which the diffusivity is time varying. If $L = h$ (Dyer and Soulsby 1988), then $\beta = w_s/(\kappa u_*)$ and this is the Rouse number of the suspension. The magnitude of the Rouse number and settling parameter β provide a measure of the ability of the flow to suspend particles. The threshold of suspension occurs when $u_*/w_s \approx 1$; smaller values of this ratio ($\beta \ll 1$) corresponding to well-mixed suspensions having relatively high loads of suspended sediment. When $\beta \gg 1$, steady suspensions feature only very low loads of suspended sediment because the flow does not generate sufficient bed shear stresses to erode and entrain particles.

The streamwise eddy diffusivity, K_x, is assumed equal to γK_z, where in the context of investigating the response of suspensions, Jobson and Sayre (1970a) and Stansby and Awang (1998) suggest $\gamma = 0$, whereas Claudin et al. (2011) set $\gamma = 1$. Further the basal condition on the concentration field introduces a mass loading parameter defined as the ratio of m_e to the settling velocity w_s, ($M = m_e/w_s$). This sets the scale of the volume fraction of sediment and influences the capacity of the flow.

By applying the dimensional scalings, the governing advection-diffusion-settling equation [i.e., Eq. (1)] is written in the dimensionless parameters listed previously

$$\frac{\partial \phi}{\partial T} + U\frac{\partial \phi}{\partial X} = \frac{\partial}{\partial Z}\left(\phi + \frac{f(Z)}{\beta}\frac{\partial \phi}{\partial Z}\right) + \Gamma\beta\frac{\partial}{\partial X}\left(f(Z)\frac{\partial \phi}{\partial X}\right) \tag{7}$$

where $\Gamma = \gamma\kappa^4 L^2/h^2$. This equation is solved on the domain $0 \leq Z \leq 1$, which is subject to the dimensionless boundary conditions given by $f(Z)$

$$\frac{\partial \phi}{\partial Z} = -Q \equiv -M\beta\begin{cases} \left(\frac{\theta}{\theta c} - 1\right)p & \theta \geq \theta c \\ 0 & \theta < \theta c \end{cases} \quad \text{at} \quad Z = 0 \tag{8}$$

$$f(Z)\frac{\partial \phi}{\partial Z} + \beta\phi = 0 \qquad \text{at} \qquad Z = 1 \tag{9}$$

Eddy Diffusivity Closures

The final requirement to complete this model is specifying the functional form of the eddy diffusivity $[f(Z)]$. The simplest eddy diffusivity model is to assume that the eddy diffusivity function is spatially invariant ($i.e., f = 1$); this implies that the effects of turbulent resuspension of particulate material is homogenous throughout the flow. This choice is popular because of its simplicity, but many alternative closures exist that capture the effects of the turbulence more accurately (Dyer and Soulsby 1988). In this paper, the empirical closures of Rouse (1938) and van Rijn (1984b) are employed, although the methods presented would work for any distribution and some of the results illustrate the invariance of this choice. The normalized Rouse and van Rijn eddy diffusivity functions are given by, respectively,

$$f = \frac{(Z+Z_0)(1-Z)}{\left(\frac{1}{2}+\frac{Z_0}{2}\right)^2} \quad \text{and} \quad f = \begin{cases} \frac{(Z+Z_0)(1-Z)}{\left(\frac{1}{2}+\frac{Z_0}{2}\right)^2} & Z \le \frac{1}{2}-\frac{Z_0}{2} \\ 1 & Z > \frac{1}{2}-\frac{Z_0}{2} \end{cases} \tag{10}$$

The authors reiterate that in the dimensionless variables introduced in this study, all closures have been normalized so the maximum of $f(Z)$ is 1. A roughness height, $Z_0 \ll 1$, has been introduced so the eddy diffusivity is strictly greater than zero at the base of the flow. The van Rijn eddy diffusivity is therefore parabolic in the region $0 \le Z \le (1-Z_0)/2$, and constant in the region $(1-Z_0)/2 < Z \le 1$. Therefore, $f(Z)$ is continuous and smooth at $Z = (1-Z_0)/2$. Dyer and Soulsby (1988) further discuss these eddy diffusivity closures and other variations. With the Rouse and van Rijn forms of $f(Z)$, the lengthscale $L = h$ and the settling parameter, β, is identical to the Rouse number; if the diffusivity is uniform, then $L = h/6$ (Claudin et al. 2011).

Unsteady and Inhomogeneous Flows

The response of the concentration field to changes in the suspending flow field or sediment supply are now examined. To this end, solutions to the governing advection-diffusion-settling Eq. (7) are analyzed when an abrupt change occurs in the conditions at $X = 0$ or $T = 0$. This could correspond to an instantaneous reduction in the bed friction velocity at $T = 0$ (generating a temporal response) or a change in sediment supply at $X = 0$ (generating a purely spatial response). Such changes alter the ability of the flow to maintain sediment in suspension and the concentration therefore progressively adjusts from an initial steady state [$\phi = \phi_I(Z)$] to a new steady state [$\phi = \phi_F(Z)$] characterised by the settling parameter and the sediment supply. For flows in which the bed friction velocity is altered, it is possible for the Shields parameter of the new state to fall below the critical value for erosion from the bed. In this case, the flow can not steadily suspend any material and so ϕ_F vanishes. By denoting the settling parameter for the initial and final states by β_I and β_F, respectively, the steady solutions of Eq. (7), dependent only on Z, are given by

$$\phi_I = \frac{Q_I}{\beta_I}\exp\left(-\int_0^Z \frac{\beta_I}{f(s)}\,\mathrm{d}s\right) \quad \text{and} \quad \phi_F = \frac{Q_F}{\beta_I}\exp\left(-\int_0^Z \frac{\beta_F}{f(s)}\,\mathrm{d}s\right) \tag{11}$$

In this expression, Q_I and Q_F denote the dimensionless fluxes of sediment from the basal boundary in the initial and final situations given by Eq. (8) and evaluated at the initial and final flow conditions, respectively.

Because the governing Eq. (7) is linear, a solution is sought in $X > 0$ for $T > 0$ of the form

$$\phi(X,Z,T) = \phi_F(Z) + \phi_D(X,Z,T) \tag{12}$$

where ϕ_D represents a temporally and spatially decaying field as the volume fraction adjusts from its initial value. Both ϕ_F and ϕ_D satisfy the same governing Eq. (7). However, the boundary conditions applicable to the decaying field are

$$f(Z)\frac{\partial \phi_D}{\partial Z} = 0 \quad \text{at} \quad Z = 0 \qquad \text{and} \qquad f(Z)\frac{\partial \phi_D}{\partial Z} + \beta_F \phi_D = 0 \quad \text{at} \quad Z = 1 \tag{13}$$

A separable solution for ϕ_D is given by

$$\phi_D = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(X) R_n(T) S_{mn}(Z) \tag{14}$$

Substituting this ansatz into the advection-diffusion-settling Eq. (7), the components $P_m(X)$ and $R_n(T)$ are

$$P_m(X) = \mathrm{e}^{-\mu_m X} \quad \text{and} \quad R_n(T) = \mathrm{e}^{-\lambda_n T} \tag{15}$$

independent of the eddy diffusivity model used. However, the eigenvalues μ_m and λ_n are dependent on the choice of model. (Here the eigenvalues are ordered so that $\mu_0 = 0 < \mu_1 < \mu_2 < \mu_3 \ldots$ and $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots$.) From Eq. (7), the component $S_{mn}(Z)$ is determined by the general eigenvalue equation

$$\frac{\partial}{\partial Z}\left(f(Z)\frac{\partial S_{mn}}{\partial Z}\right) + \beta_F \frac{\partial S_{mn}}{\partial Z} + (\lambda_n \beta_F + \mu_m U \beta_F + \beta_F^2 \Gamma \mu_m^2 f) S_{mn} = 0 \tag{16}$$

subject to the homogeneous boundary conditions (13). In the problems that follow, the authors are interested in either purely temporal or purely spatial decay toward the final steady state. These correspond to the two families of functions, $S_{0n}(Z)$ and $S_{m0}(Z)$, in which $S_{00}(Z) = 0$. It is then possible to identify the dominant length or time scales of the decaying solution, ϕ_D. These are given in the smallest nonzero eigenvalues λ_1 and μ_1 (Pritchard 2006). The dimensional length and time scales of the response, x_r and t_r, are therefore given as

$$x_r = \frac{h u_*}{\kappa w_s \mu_1} \quad \text{and} \quad t_r = \frac{h}{w_s \lambda_1} \tag{17}$$

In what follows solutions are analyzed first for the constant diffusivity before tackling numerically the results for the Rouse and van Rijn diffusivities. Focus is on computing the timescale on which the suspension responds and it is demonstrated that this is independent of the form of the diffusivity in the regimes $\beta_F \ll 1$ and $\beta_F \gg 1$.

Constant Eddy Diffusivity

For a constant diffusivity [i.e., $f(Z) = 1$], the concentration profile changes from an initial to final state, given by, respectively,

$$\phi_I = M\left(\frac{\theta_I}{\theta_c} - 1\right)^p e^{-\beta_I Z} \quad \text{and} \quad \phi_F = M\left(\frac{\theta_F}{\theta_c} - 1\right)^p e^{-\beta_F Z} \tag{18}$$

where θ_I and θ_F denote the values of the Shields parameter in the initial and final states.

The purely temporal case for which the decaying volume fraction was derived by Tu et al. (1993) and Pritchard (2006) is given by

$$S_{0n}(Z) = B_n e^{-\frac{\beta_F}{2}Z}\left(\frac{2\alpha_n}{\beta_F}\cos(\alpha_n Z) + \sin(\alpha_n Z)\right), \tag{19}$$

where B_n are constants determined by the initial condition and the eigenvalue λ_n is determined by

$$\lambda_n = \frac{\beta_F}{4} + \frac{\alpha_n^2}{\beta_F} \quad \text{and} \quad \tan\alpha_n = \frac{4\alpha_n \beta_F}{4\alpha_n^2 - \beta_F^2} \tag{20}$$

Fig. 1 plots the variation of the smallest eigenvalue, λ_1, as a function of β_F. Following Pritchard (2006), it is possible to show that the asymptotic behavior in the regime $\beta_F \ll 1$

$$\lambda_1 = 1 + \frac{\beta_F}{6} + \dots \tag{21}$$

and in the regime $\beta_F \gg 1$

$$\lambda_1 = \frac{\beta_F}{4} + \frac{\pi^2}{\beta_F} + \dots \tag{22}$$

These asymptotic estimates (Fig. 1) capture accurately the dependence of the smallest eigenvalue λ_1 on the settling parameter β_F in their appropriate regimes.

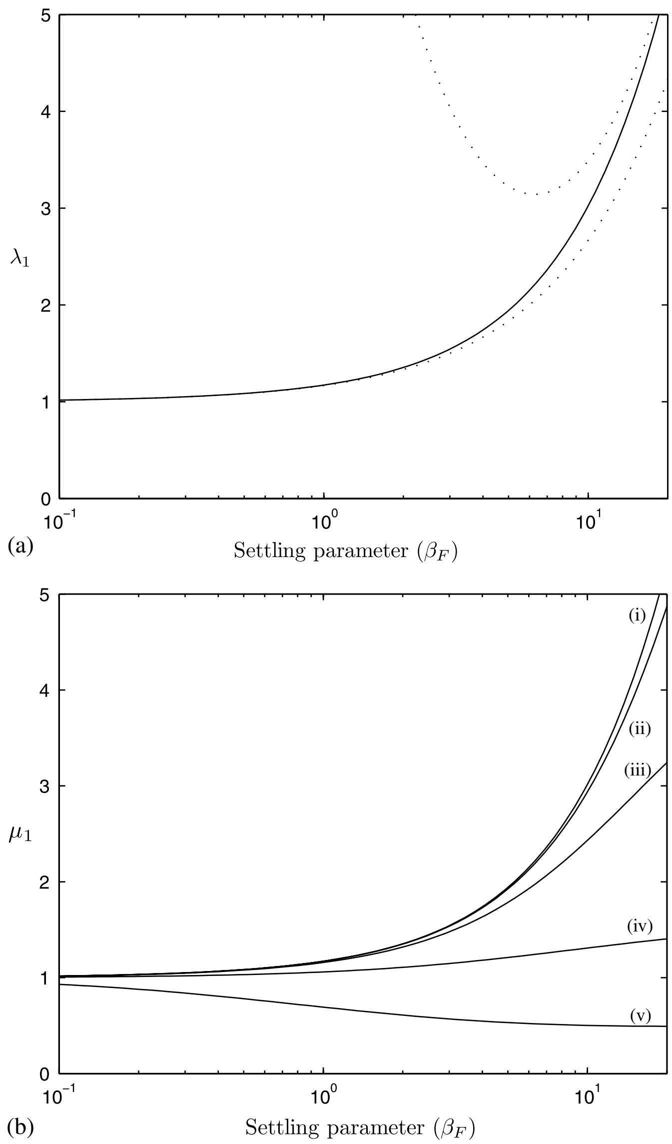

Fig. 1. The smallest eigenvalue (λ_1 or μ_1) as a function of the settling parameter when the diffusivity is constant ($f(Z) = 1$) for (a) the purely temporal problem and (b) the purely spatial problem; In (a) the solid line corresponds to the exactly computed solution and the dashed lines correspond to the asymptotic forms in the regimes $\beta_F \ll 1$ and $\beta_F \gg 1$; In (b) the solid curves (i)–(v) correspond to the following values of Γ: (i) $\Gamma = 0$; (ii) $\Gamma = 0.001$; (iii) $\Gamma = 0.01$; (iv) $\Gamma = 0.1$; and (v) $\Gamma = 1$

The spatial problem is analyzed next under the further assumption that the velocity field is spatially uniform (Stansby and Awang 1998; Mei 1969). It is found that

$$U\mu_n + \Gamma\beta_F\mu_n^2 = \frac{\beta_F}{4} + \frac{\alpha_n^2}{\beta_F} \quad \text{and} \quad \tan\alpha_n = \frac{4\alpha_n\beta_F}{4\alpha_n^2 - \beta_F^2} \tag{23}$$

Fig. 1 plots the dependence of the smallest eigenvalue, μ_1, for various values of Γ. It is also possible to readily identify the asymptotic behaviors in the regime $\beta_F \ll 1$

$$\mu_1 = 1/U + \left(\frac{1}{6U} - \frac{\Gamma}{U^3}\right)\beta_F + \dots \tag{24}$$

whereas in the regime $2U/\sqrt{\Gamma} \gg \beta_F \gg 1$

$$\mu_1 = \frac{\beta_F}{4U} - \frac{\pi^2}{U\beta_F} + \dots \tag{25}$$

and finally when $\beta_F \gg 2U/\sqrt{\Gamma}$

$$\mu_1 = \frac{1}{2\sqrt{\Gamma}} - \frac{U}{2\Gamma\beta_F} + \dots \tag{26}$$

The regime of $\beta_F \gg 1$ is not realized by erosional flows for which the Shields number does not exceed the critical value for incipient motion. However, this regime ($\beta_F \gg 1$) could be accessed by releasing suspended sediment into the flow and studying how it settles out of suspension (see the discussion of experimental studies in the section "Comparison to Experimental Results"). Claudin et al. (2011) also calculated the lengthscales of response when the diffusivity was spatially uniform, establishing the asymptotic result [Eq. (24)] in the regime $\beta_F \ll 1$. Claudin et al. (2011) also suggested that essentially the same results arose if the velocity field were logarithmic ($U = \log(1 + z/z_0)$) instead of pluglike. This observation may be demonstrated numerically by calculating the smallest eigenvalue, μ_1 as a function of β_F. In Fig. 2, the eigenvalue μ_1, normalized by the average velocity ($\int_0^1 U \mathrm{d}Z$), is plotted as a function of β_F for a uniform and logothrimic velocity profile and close correspondence is shown between the two cases. In fact, this correspondence is a manifestation of much more general result that is subsequently established.

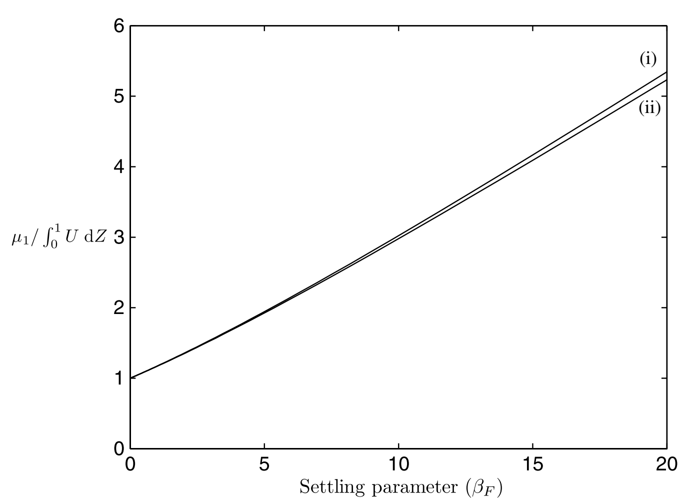

Fig. 2. The smallest eigenvalue for the purely spatial problem, μ_1, normalized by the dimensionless average velocity, $\int_0^1 U(Z)\mathrm{d}Z$, and plotted as a function of the settling parameter β_F when $f(Z) = 1$ for (i) $U = 1$ and (ii) $U = \log(1 + Z/Z_0)$ with $Z_0 = 10^{-6}$

Spatially Varying Eddy Diffusivities

The time- and lengthscales of response of suspensions are now analyzed by using a spatially varying diffusivity $f(Z)$ with a particular focus on the Rouse and van Rijn distributions introduced previously in Eq. (10), when the dimensionless velocity field is $U = \log(1 + Z/Z_0)$. By using a boundary value solver in MatLab, the authors numerically integrated the differential Eq. (16) subject to (13) to determine the eigenvalues, λ_n and μ_m. Some care is required when using the Rouse diffusivity, which vanishes at $Z = 1$. However, this can be tackled by calculating the local expansion of $S(Z)$ in this region. In these calculations, the dimensionless roughness is given by $Z_0 = 10^{-6}$; however, similar behavior occurs for other values. In Figs. 3 and 4, the smallest eigenvalue, λ_1 or μ_1, is plotted as a function of the settling parameter, β_F, which is identical to the Rouse number for these diffusivities. Eigenvalues λ_1 and μ_1 are increasing functions of the Rouse number; that they attain values for relatively small β_F, which are independent of the assumed form of the diffusivity and of Γ; and that their values for relatively large values of β_F are dependent on Γ.

The authors then determined the asymptotic behavior of the eigenvalues λ_1 and μ_1 and demonstrate that its value in the regimes $\beta_F \ll 1$ and $\beta_F \gg 1$ is independent of $f(Z)$.

Asymptotic Solutions for the Regime $\beta_F \ll 1$

The smallest nonzero eigenvalues, μ_1 and λ_1 in the regime $\beta_F \ll 1$ may be evaluated by determining the asymptotic forms of the solution for $S_{01}(Z)$ and λ_1 and for $S_{10}(Z)$ and μ_1 in Eq. (16). First, the temporal problem is treated by posing the following series expansions:

$$\phi_D = e^{-\lambda_1 T}(\tilde{S}_0(Z) + \beta_F \tilde{S}_1(Z) + \beta_F^2 \tilde{S}_2(Z) + \dots) \tag{27}$$

$$\lambda_1 = \lambda_{10} + \beta_F \lambda_{11} + \beta_F^2 \lambda_{12} + \dots \tag{28}$$

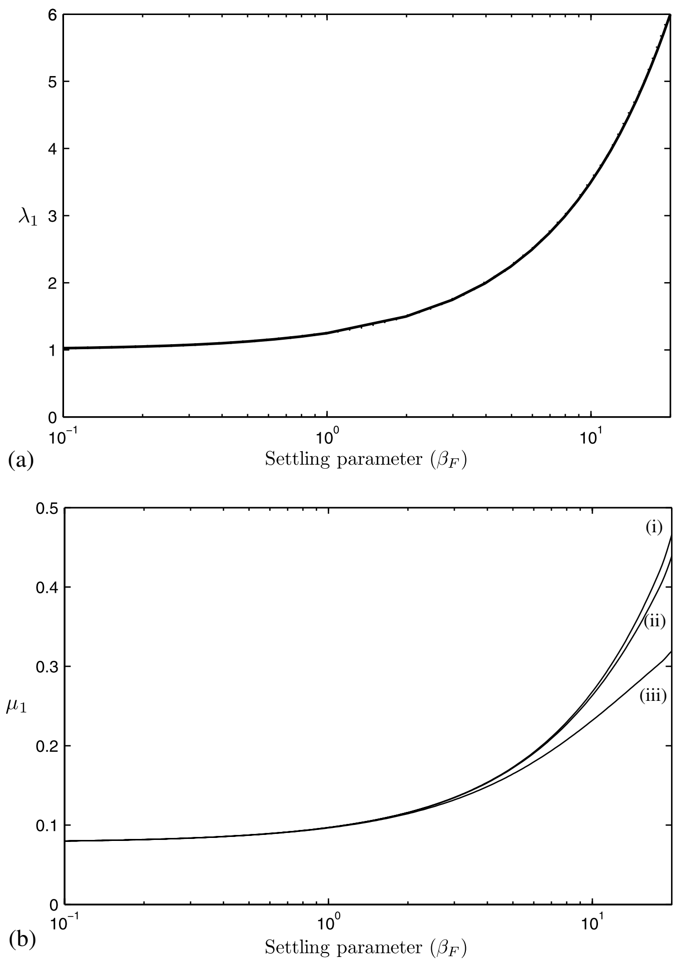

Fig. 3. The smallest eigenvalue (λ_1 or μ_1) as a function of the settling parameter when the diffusivity is given by the Rouse form for (a) the purely temporal problem (λ_1) and (b) the purely spatial problem (μ_1); In (a) the solid line corresponds to the exactly computed solution and the dashed lines (indistinguishable) correspond to the asymptotic forms in the regimes $\beta_F \ll 1$ $(\lambda_1 = 1 + \beta_F \lambda_{11} + \dots)$ and $\beta_F \gg 1$ $(\lambda_1 = \beta_F/4 + 1/(1 + Z_0) + \dots)$; In (b) the solid curves correspond to the following values of Γ: (i) $\Gamma = 0$; (ii) $\Gamma = 10^{-1}$; and (iii) $\Gamma = 1$

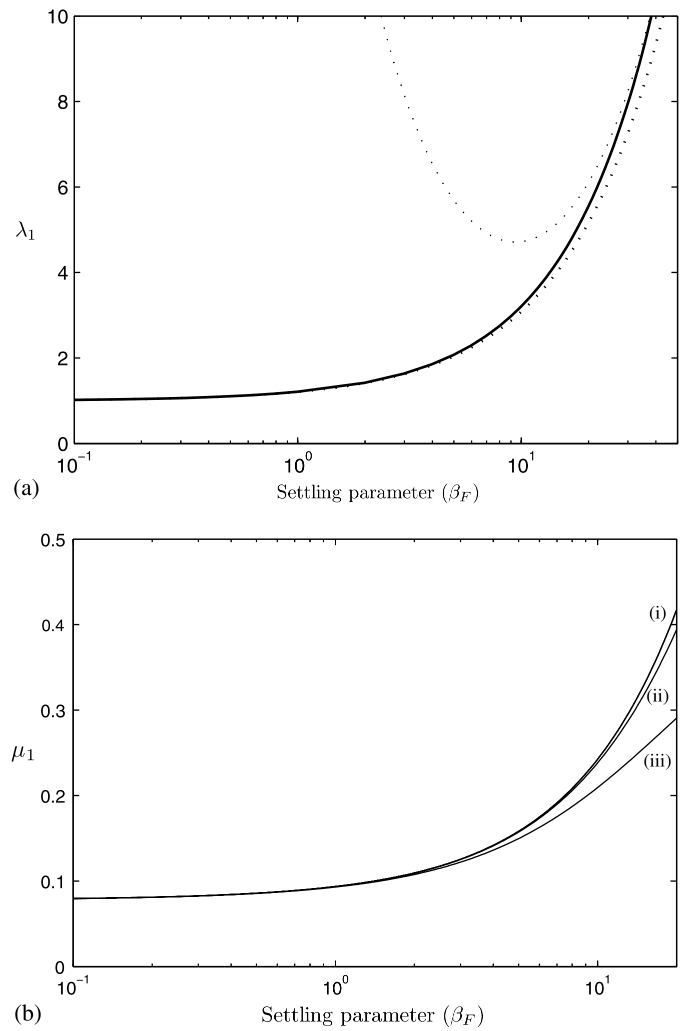

Fig. 4. The smallest eigenvalue (λ_1 or μ_1) as a function of the settling parameter when the diffusivity is given by the van Rijn form for (a) the purely temporal problem (λ_1) and (b) the purely spatial problem (μ_1); In (a) the solid line corresponds to the exactly computed solution and the dashed lines correspond to the asymptotic forms in the regimes $\beta_F \ll 1$ $(\lambda_1 = 1 + \beta_F \lambda_{11} + \dots)$ and $\beta_F \gg 1$ $(\lambda_1 = \beta_F/4 + 9\pi^2/[4(1 + Z_0)^2 \beta_F] + \dots)$; In (b) the solid curves correspond to the following values of Γ: (i) $\Gamma = 0$; (ii) $\Gamma = 10^{-1}$; and (iii) $\Gamma = 1$

for the unknown functions $\tilde{S}_i(Z)$ and constants λ_{1i}. These are substituted into the governing equations and equated at successive powers of β_F. At $O(1)$, it is found that

$$\frac{d}{dZ}\left(f(Z)\frac{d\tilde{S}_0}{dZ}\right) = 0, \quad \text{subject to } f\frac{d\tilde{S}_0}{dZ} = 0 \quad \text{at} \quad Z = 0, 1 \tag{29}$$

Therefore, $\tilde{S}_0 = C$, a constant that is not yet undetermined. At $O(\beta_F)$,

$$\frac{d}{dZ}\left(f(Z)\frac{d\tilde{S}_1}{dZ}\right) + \frac{d\tilde{S}_0}{dZ} + \lambda_{10}\tilde{S}_0 = 0 \tag{30}$$

which is subject to $f d\tilde{S}_1/dZ + \tilde{S}_0 = 0$ at $Z = 1$ and $f d\tilde{S}_1/dZ = 0$ at $Z = 0$. Therefore, it is deduced that

$$\tilde{S}_1 = C_1 - C\int_0^Z \frac{Z'}{f(Z')}dZ \quad \text{and} \quad \lambda_{10} = 1 \tag{31}$$

where C_1 is another constant that is not yet undetermined. At $O(\beta_F^2)$, the authors found

$$\frac{d}{dZ}\left(f(Z)\frac{d\tilde{S}_2}{dZ}\right) + \frac{d\tilde{S}_1}{dZ} + \lambda_{10}\tilde{S}_1 + \lambda_{11}\tilde{S}_0 = 0 \tag{32}$$

which is subject to $f\partial\tilde{S}_2/\partial Z + \tilde{S}_1 = 0$ at $Z = 1$ and $f\partial\tilde{S}_2/\partial Z = 0$ at $Z = 0$. Therefore, it is deduced that

$$\lambda_{11} = \int_0^1\int_0^Z \frac{Z'}{f(Z')}dZ'dZ = \int_0^1 \frac{Z'(1-Z')}{f(Z')}dZ' \tag{33}$$

In dimensional variables, the timescale for response is given by

$$t_r = \frac{h}{w_s}(1 - \beta_F\lambda_{11} + \mathcal{O}(\beta_F^2)), \quad \beta_F \ll 1 \tag{34}$$

At the leading order, the timescale is independent of the form of the eddy diffusivity $f(Z)$, which influences λ_1 only at $O(\beta_F)$. This implies that at small values of the settling parameter (when the intensity of turbulence is relatively high), the timescale of response to change in flow conditions is given by the relatively slow timescale on the basis of settling through the flow depth (h/w_s).

The analysis for the purely spatial problem follows analogous steps. The volume fraction is posed as $\phi_D = e^{-\mu_1 X}(\hat{S}_0 + \beta_F\hat{S}_1 + \ldots)$, the eigenvalue $\mu_1 = \mu_{10} + \beta_F\mu_{11} + \ldots$ and deduced that

$$\mu_1 = \frac{1}{\int_0^1 U(s)ds} + O(\beta_F) \quad \text{when } \beta_F \ll 1 \tag{35}$$

In the dimensional form, the leading order expression for the lengthscale of response is given by

$$x_r = \frac{hu_*}{\kappa w_s}\int_0^1 U(s)ds = \frac{h\overline{u}}{w_s} \tag{36}$$

where $\overline{u}$ denotes the depth-averaged flow speed. This result for the lengthscale of response in the regime $\beta_F \ll 1$ is independent of the form of the eddy diffusivity.

Asymptotic Solutions for the Regime $\beta_F \gg 1$

The asymptotic solutions for the leading order decay rates λ_1 and μ_1 may be determined in the regime $\beta_F \gg 1$. In this section, a general method is presented for calculating the dependence of λ_1 and μ_1 upon β_F, which is illustrated for each of the three forms of diffusivity that have been analyzed throughout this paper. However, the analysis also establishes that the leading order dependence of the decay rate is independent of the functional form of the diffusivity.

For the first step, the decaying component of the concentration field for the purely temporal and spatial problem is written as

$$\left\{\begin{matrix}\phi_D(Z,T)\\ \phi_D(X,Z)\end{matrix}\right\} = \sum_{n=1}^{\infty}\left\{\begin{matrix}e^{-\lambda_n T}\\ e^{-\mu_n X}\end{matrix}\right\} f(Z)^{-1/2} \times \exp\left(-\int_0^Z \frac{\beta_F}{2f(Z')}dZ'\right)\psi_n(Z) \tag{37}$$

This transforms the governing equation to

$$\frac{d^2\psi_n}{dZ^2} + \beta_F^2 H_n(Z)\psi_n = 0 \tag{38}$$

where

$$H_n(Z) = \frac{\chi_n}{\beta_F f} - \frac{1}{4f^2} + \frac{f'^2}{4\beta_F^2 f^2} - \frac{f''}{2\beta_F^2 f} \tag{39}$$

and $\chi_n = \lambda_n$ or $\chi_n = \mu_n U + \mu_n^2\Gamma\beta_F f$ for purely temporal and spatial problems, respectively. In this expression, a prime denotes differentiation with respect to Z. The boundary conditions become

$$\begin{aligned} f\psi_n' - \tfrac{1}{2}(\beta_F + f')\psi_n = 0 \quad \text{at} \quad Z = 0 \quad \text{and} \\ f\psi_n' + \tfrac{1}{2}(\beta_F - f')\psi_n = 0 \quad \text{at} \quad Z = 1 \end{aligned} \tag{40}$$

In the regime $\beta_F \gg 1$, this problem is now amenable to WKB techniques (Hinch (1992) for which the asymptotic solution is given by

$$\psi_n(Z) = \frac{1}{H_n^{1/4}}\left(C\cos\left(\beta_F\int^Z H_n^{1/2}dZ'\right) + D\sin\left(\beta_F\int^Z H_n^{1/2}dZ\right)\right) \tag{41}$$

in the regime $H_n(Z) > 0$, with C and D constants that are not yet undetermined. If conversely $H_n(Z) < 0$, then a similar expression to Eq. (14) exists, but hyperbolic functions replace the trigonometric functions. To the leading order, the boundary conditions demand that $\psi_n(0) = \psi_n(1) = 0$; therefore, if $H_n(Z) < 0$ for $0 \le Z \le 1$, then only the trivial solution $\psi_n(Z) = 0$ is found. However, if $H_n(Z) > 0$ for $0 \le Z \le 1$, then the boundary conditions may be satisfied if

$$\sin\left(\beta_F\int_0^1 H_n^{1/2}dZ\right) = 0 \tag{42}$$

Such a regime occurs for the temporal problem with a constant diffusivity *i.e.*, $f = 1$, in which case the leading order eigenvalue equation for λ_n is given by

$$\beta_F\left(\frac{\lambda_n}{\beta_F} - \frac{1}{4}\right)^{1/2} = n\pi, \quad n \in \mathbb{Z} \tag{43}$$

In the regime $\beta_F \gg 1$, this leads to $\lambda_n = \frac{1}{4}\beta_F + \frac{n^2\pi^2}{\beta_F} + \ldots$. This result reproduces exactly the first two terms of the asymptotic expression derived previously [Eq. (22)].

However, the function $H_n(Z)$ will generally have regions in which it is positive or negative, as in the Rouse and van Rijn distributions introduced previously in Eq. (10). In this case, applying the WKB method is more complicated because of the form of the function $H_n(Z)$ as it passes through zero. In the case in which two zeros exist for $H_n(Z)$ so that $H_n(Z) > 0$ for $Z_2 > Z > Z_1$ (as for the Rouse diffusivity), then the eigenvalue equation becomes [Hinch (1992)]

$$\sin\left(\beta_F \int_{Z_1}^{Z_2} H_n^{1/2} dZ + \frac{\pi}{2}\right) = 0 \qquad (44)$$

However, if just a single zero exists so that $H_n(Z) > 0$ for $1 \geq Z > Z_1$ (as for the van Rijn diffusivity), then the eigenvalue equation becomes [Hinch (1992)]

$$\sin\left(\beta_F \int_{Z_1}^{1} H_n^{1/2} dZ + \frac{\pi}{4}\right) = 0 \qquad (45)$$

The eigenvalue may then be evaluated in the regime $\beta_F \gg 1$. For Eq. (44) or Eq. (45) to remain balanced require that $\max(\chi_n f) = \beta_F/4$. The leading order eigenvalues can immediately be deduced for the purely temporal problem

$$\lambda_1 = \frac{\beta_F}{4} + \ldots \quad \text{when } \beta_F \gg 1 \qquad (46)$$

For the purely spatial problem when $\Gamma \neq 0$, then

$$\mu_1 = \frac{1}{2\sqrt{\Gamma}} + \ldots \quad \text{when } \beta_F \gg 1 \qquad (47)$$

Both of these results are independent of the form of the eddy diffusivity. When $\Gamma = 0$ (Stansby and Awang 1998), then

$$\mu_1 = \frac{\beta_F}{4\max(fU)} + \ldots \quad \text{when } \beta_F \gg 1 \qquad (48)$$

In this latter case, a weak dependence exists on the form of the diffusivity. For the Rouse diffusivity with $Z_0 = 10^{-6}$. It may be calculated that $\max(fU) = 13.14$, whereas for the van Rijn diffusivity, the authors calculated $\max(fU) = 13.82$.

These results may be established for a general diffusivity $f(Z)$ by using this WKB analysis. To this end, it is assumed that m regions exist in which $H_n(Z) > 0$ with $H_n(Z_{2j-1}) = 0$ and $H_n(Z_{2j}) = 0$ $(1 \leq j \leq m)$. The WKB eigenvalue equation then becomes

$$\sin\left(\beta_F \sum_{j=1}^{m}\left(\int_{Z_{2j-1}}^{Z_{2j}} H_n^{1/2} dZ + \frac{\pi}{2}\right)\right) = 0 \qquad (49)$$

In the regime $\beta_F \gg 1$, it may be deduced that Eq. (49) can only remain balanced if $\max(\chi_n f) = \beta_F/4$, as previously. Therefore, for the purely temporal problem, the dimensional timescale of response is given by

$$t_r = \frac{4\kappa u_* L}{w_s^2} \qquad (50)$$

whereas for the purely spatial problem, the dimensional lengthscale of response is

$$x_r = \frac{2\sqrt{\gamma}\,\kappa u_* L}{w_s} \quad (\gamma \neq 0) \qquad x_r = \frac{4u_*^2 L}{w_s^2}\max(fU) \quad (\gamma = 0) \qquad (51)$$

Comparison To Experimental Results

Several experimental studies have measured the spatial development of sediment suspensions in response to a sustained change of boundary conditions and the results of this paper may be compared with these observations. The experiments encompass flows that are either net depositional (Jobson and Sayre 1970b; Ashida and Okabe 1982; Celic and Rodi 1988) in which an excess of sediment is supplied at $x = 0$ and evolves downstream to a steady state or they are net erosional (Ashida and Okabe 1982; van Rijn 1984a) in which an initially sediment-free fluid encounters an erosional bed at $x = 0$ and picks up sediment until attaining the steady state. The former flow may be termed "overcapacity" and deposit until attaining equilibrium, whereas the latter are "undercapacity" and erode until attaining equilibrium. The experimental studies report the suspended sediment flux at various locations downstream of the point at which sediment is introduced to the flow, crucially over relatively long distances so that the approach to the downstream equilibrium may be assessed.

The measured spatial decay of the suspensions toward the steady state may be compared with the leading order decay rates predicted by the theoretical calculations. To this end, the leading order decay rate, μ_1, is calculated for a given settling parameter. In Fig. 5, the predicted lengthscale of response relative to the flow depth, x_r/h, is plotted as a function of $w_s/[\kappa u_*]$, when the dimensionless roughness length $Z_0 = 10^{-6}$, the dimensionless average flow speed is 12.8, and $\gamma = 1$, for each of the forms of the diffusivity investigated in detail in this study. The results for the van Rijn and Rouse forms are very similar, but the results for the uniform diffusivity always exceed the predictions for the other two diffusivity forms by a factor of up to approximately 2 in the range plotted. The van Rijn model of the diffusivity is selected to compare the theoretical predictions with the experimental data and the smallest eigenvalue, μ_1, is calculated with the further assumption that the velocity field is logarithmic [$U = \log(1 + z/z_0)$] and that streamwise diffusion is included ($\gamma = 1$); however, this latter factor plays only a negligible effect on the mangnitube of μ_1. This indicates that the attempts to fit the experimental data differ from the results of Claudin et al. (2011), who employed a uniform diffusivity. All erosional and depositional experiments report the bed friction velocity, u_*, but not all experiments provide the roughness length, which is required to determine the velocity profile. This may be determined by imposing a logarithmic dimensionless velocity profile, $U(z) = \log(1 + z/z_0)$, and requiring that $\int_0^1 U(z)dz$ gives the reported dimensionless mean velocity.

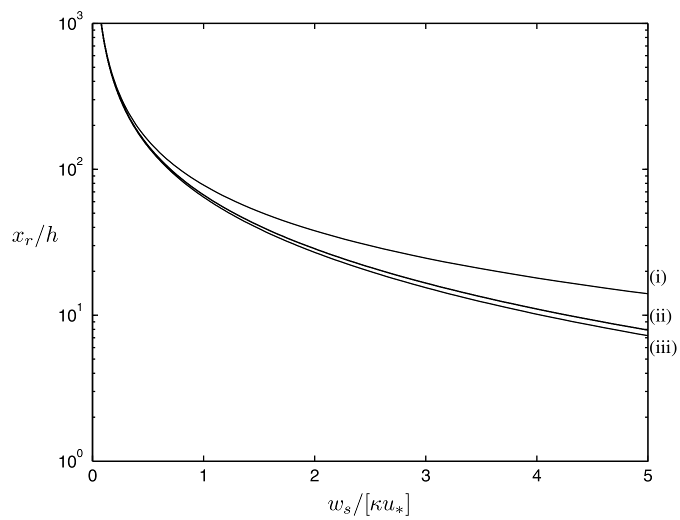

Fig. 5. The lengthscale of response, relative to the depth of the flow (x_r/h), as a function of $w_s/[\kappa u_*]$ with dimensionless roughness length $Z_0 = 10^{-6}$, dimensionless average velocity $\overline{u}\kappa/u_* = 12.8$, and $\gamma = 1$ for (i) uniform eddy diffusivity; (ii) van Rijn diffusivity; and (iii) Rouse diffusivity

For the depositional studies, the difference between the measured downstream flux of sediment, $q_s(x)$, and its downstream value, q_∞, is plotted as a function of the downstream distance. This is normalized by the difference between the initial value and q_∞, $\Delta q_d = [q_s(x) - q_\infty]/(q_s(0) - q_\infty)$ [Fig. 6(a)]. In dimensional variables, the flux is defined by

$$q_s(x) = \int_0^h cu\,\mathrm{d}z. \tag{52}$$

Jobson and Sayre (1970b) also presented data for the spatial evolution of the normalized cumulative volume of deposited particles as a function of distance, denoted by $1 - \Phi(x)$, and measured with much greater spatial resolution than the suspended flux of particles. In Fig. 6, $\Phi(x)$ is plotted as a function of distance as a further test of the theoretical predictions because this quantity will also exhibit the same e-folding lengthscales. Good agreement exists between the experimental observations and the theoretical predictions [see Fig. 6(a) and Table 1].

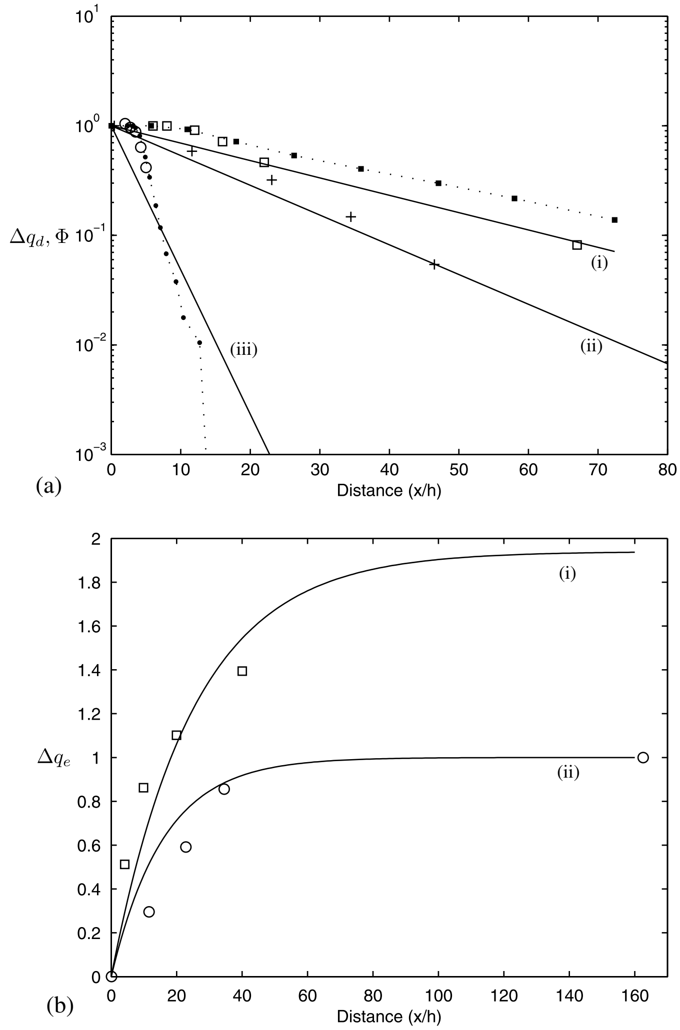

Fig. 6. Comparison of the theoretical predictions and the experimental measurements for (a) depositional experiments and (b) erosional experiments; In (a) the symbols correspond to the direct measurements of the relative sediment flux (Δq_d: Φ: series FS1, Jobson and Sayre (1970b); □: series CS1, Jobson and Sayre (1970b); and +: Ashida and Okabe (1982)]; the connected symbols correspond to the measurements of the cumulative deposit, Φ; and the solid lines correspond to the exponential decay corresponding to the smallest eigenvalue; In (b) the symbols correspond to direct measurements of the relative sediment flux (Δq_e: □: van Rijn (1984a); ∘ Ashida and Okabe (1982; and the solid lines correspond to the exponential increase $C[1 - \exp(-\mu_1 X)]$ associated with the smallest eigenvalue, where $C = 1$ for the Ashida and Okabe (1982) data and C takes fitted value of 1.94 for the van Rijn (1984a) data

Table 1. Experimental Conditions, Parameter Values and Calculated Eigenvalues

Experiment	u_*	w_s	$w_s/[\kappa u_*]$	z_0/h	μ_1	x_r/h
Jobson and Sayre (1970b) Data series FS1	4.48	1.05	0.59	0.023	0.386	27.6
Jobson and Sayre (1970b) Data series CSI	4.51	6.2	3.44	0.023	0.551	3.30
Ashida and Okabe (1982)	3.54	0.019	1.31	0.006	0.298	16.0
van Rijn (1984a)	4.8	2.2	1.15	0.0013	0.215	25.2

For the erosional flows, the suspended flux normalized by the equilibrium flux far downstream $\Delta q_e = q_s(x)/q_\infty$ [Fig. 6(b)] is plotted as a function of distance downstream. The experiments conducted by Ashida and Okabe (1982) have measurements sufficiently far downstream that the quantity, q_∞, may be readily determined; however, for van Rijn (1984a) some uncertainty exists in this equilibrium value. The value of q_∞ theoretically calculated by van Rijn (1984a) is in fact exceeded by the measured fluxes. Several possible reasons may exist for this difference. For example, the theoretical prediction uses an empirical formula that may be subject to significant error and the erosive flows generate a scour pit, which is not included in these computations (van Rijn 1984a). With these uncertainties, the authors determined a best fit for the flux far downstream. Reasonable agreement exists between the experimental measurements and the theoretical predictions [see Fig. 6(b) and Table 1].

Summary and Conclusions

In this paper, the unsteady response of suspensions to instantaneous changes in their ability to support material and the spatial evolution when the sustained supply of sediment is altered has been analyzed. The turbulence has been modeled by using a simple empirical closure in which its effect on the relatively dense sediment is captured through eddy diffusivity. Many closures have been suggested for this type [Dyer and Soulsby (1988)]. However, results presented in this paper have utilized three popular forms and established some general results. For the dimensional timescale of response, t_r, it has been shown that when the settling parameter is small ($\beta \ll 1$), $t_r = h/w_s + \ldots$ and when the settling parameter is large ($\beta \gg 1$), then $t_r = 4\kappa u_* L/w_s^2 + \ldots$. Pritchard (2006) had previously demonstrated these dependencies for a constant diffusivity. However, this paper has extended them to all other forms of diffusivity. The dimensional factors in these results may be rationalized as follows: when the settling parameter is small, the sediment is well-mixed throughout the fluid depth. Any adjustment therefore requires settling throughout the depth, which occurs on a timescale h/w_s. Conversely, when the settling parameter is large, the sediment is suspended in a boundary layer close to the basal boundary. The size of this boundary layer is K_m/w_s. Temporal adjustment requires settling over this lengthscale and this occurs on a timescale given by K_m/w_s^2. Focusing next on the spatially evolving problem, when the settling parameter is small ($\beta_F \ll 1$), the lengthscale of response is $\overline{u}t_r = \overline{u}h/w_s$ because the sediment has to settle through the entire water column. However, if the settling parameter is large ($\beta_F \gg 1$), streamwise diffusion dominates streamwise advection and the lengthscale of response is equal to the size of the vertical boundary layer, and scaled by the square root of the ratio of the vertical and streamwise diffusivities, $x_r \sim \sqrt{\gamma}K_m/w_s$.

An important consequence of this analysis is the variation of the timescale of response have different settling velocities. Provided the concentration is sufficiently dilute, the behavior of each class of particles may be considered independently and then the results derived above imply different dependencies of the response timescale upon the settling velocity, depending on whether the settling parameter for the particular class of particles of interest is much greater than or much less than unity. For polydisperse suspensions of sediment in a waning turbulent flow in which the particles are progressively settling out to the underlying boundary, the different time- and lengthscales for each class will affect the composition of the underlying deposit. Sumner et al. (2008) recently examined flows of this type experimentally. They analyzed the deposit formed under a decelerating flow and they particularly focused on the variation of the mean grain size with depth through the deposit. The pattern of grading that they observed may be partly attributed to variations in the response timescale of the particles in suspension as the flow conditions change; however, direct comparison of the experimental results and the theoretical predictions developed in this paper is not possible because the flow speed (and turbulence intensity) changed progressively, rather than instantaneously.

In conclusion, the results in this study have established that some differences exist in the predictions for suspensions when using different forms for the eddy diffusivity; however, features exist, particularly in the time- and lengthscales of response, that are invariant to the assumed form of the diffusivity in the regimes of large and small settling parameter. This suggests that when modeling the response of turbulent suspensions in complex unsteady geomorphological or sedimentological problems (Huijts et al. 2006), it may be possible to use the simplest closures for eddy diffusivity if the focus is on the generic evolution of the suspension rather than on the quantitative prediction of rates of erosion and deposition.

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