Two-dimensional granular slumps down slopes

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The slumping and subsequent arrest of initially motionless granular materials from behind a rapidly removed lockgate in a sloping two-dimensional channel is considered theoretically and experimentally. The theory is based upon a shallow layer description of the flow and arrest of the grains in which resistance to the downslope motion is modelled as a Coulomb drag with a constant coefficient of friction. The flows leave a thin layer of deposited material along the chute and the depth of the deposit at the rear of the lock is predicted from the theoretical model using asymptotic techniques. This analysis explains the dependence on the initial aspect ratio of the release that has been seen in previous numerical and experimental studies of granular slumps over horizontal surfaces. The theoretical predictions of this depth are also compared with laboratory observations of the slumping of four dry granular materials. It is shown that there is quantitative agreement between the experimental measurements and the theoretical predictions, which include no fitting parameters. The theoretical predictions for the length along the chute that the materials slump, however, are not in agreement with the theoretical model and potential reasons for this mismatch are discussed. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2762254]

I. INTRODUCTION

Recently there has been considerable scientific interest in the slumping of granular materials; initially motionless grains are released by the rapid removal of a barrier, behind which they were confined, and allowed to flow over a horizontal surface until they are fully arrested, leading to a deposit distributed over the underlying surface. The dynamics of this type of flow has been studied experimentally in an axisymmetric geometry, in which the grains are initially confined behind a cylindrical container[1,2] and in a two-dimensional channel in which they are initially behind a dam spanning the width of the channel.[3–6] These experimental studies have employed a variety of essentially noncohesive materials and have allowed the grains to flow out over smooth and roughened horizontal surfaces. These studies share the common feature that the maximum distance travelled by the grains, denoted by x_∞ and termed the runout distance, and the maximum depth of the deposited grains, h_∞, both measured relative to the initial width of the dam (or radius of the cylinder, for axisymmetric releases) depend strongly on the initial aspect ratio of the release $a \equiv h_0/x_0$, where x_0 and h_0 are the initial width and height of the release, respectively (see Fig. 1). Each of the studies suggest slightly different empirical relationships between h_∞/x_0 and x_∞/x_0 and a. Furthermore Lube *et al.*[1] found no dependence of these ratios of length scales on the type of particle, whereas Balmforth and Kerswell[3] and Lajeuness *et al.*[4] indicate that there is a weak dependence—weak, because the coefficient of friction only varies slightly for each of the materials used.

Mathematical modelling of these flows has proven to be a significant challenge. To date two scientific approaches have been employed: numerical simulations of the collapse by the use of discrete element models, in which the motion of and interaction between individual grains is explicitly calculated;[7,8] and the use of continuum models, often simplified to capture the motion of a shallow layer of grains in which vertical accelerations are negligible and the pressure is essentially hydrostatic.[9–11]

Using the former of these approaches, both Zenit[7] and Staron and Hinch[8] study the motion of a polydisperse ensemble of circular disks, slumping in two dimensions under gravity. Although these studies are idealized representations of the true experimental configurations, these simulations offers the particular advantage that the velocities and positions are precisely known at all times. Staron and Hinch[8] show how initial potential energy is converted into kinetic energy and then dissipated through particulate interactions. They were able to reproduce qualitative features observed in the experiments, to reveal relationships between the length scales of deposit (x_∞ and h_∞) and the aspect ratio a that were consistent with the measurements and they concluded that the runout distance depended strongly on the dynamics of the initial vertical motion of the collapse, as well as the basal friction.

In the latter approach to modelling, Kerswell[9] and Mangeney-Castelnau *et al.*[10] assumed that the main resistance to the flow is due to a Coulomb-type drag force, arising from the material sliding and rolling across the lower boundary. However the magnitudes of the coefficients of friction required to achieve quantitative agreement between these models and the experimental measurements of flow depths and runout lengths exceed those independently measured for the materials. These continuum models did reveal numerically an intriguing scaling relationship between the maximum depth of the deposit, h_∞/x_0, and the initial aspect ratio a; this was not of a similarity form and could not be explained through simple reasoning. In particular, Kerswell[9]

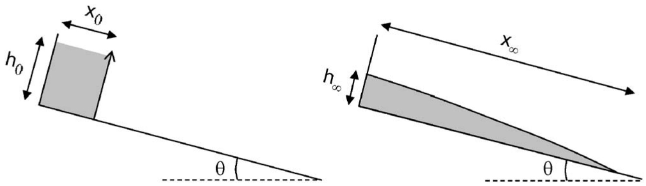

FIG. 1. The configuration of the flow. Granular material is released from rest behind a rapidly removed lock gate, flows down an inclined plane and arrests to form a deposit. The initial height of material within the lock and its length are denoted by h_0 and x_0, respectively, while the thickness of the deposit at the rear wall of the lock and the runout length are denoted by h_∞ and x_∞, respectively.

showed numerically that $h_\infty/x_0 \sim a^{1/3}$ when $a \gg 1$ and this is in accord with some of the experimental observations.[3,4] Adopting a shallow layer model may be problematic for these flows in general, because it is clear that during the initial phases of the collapse they are far from "shallow," especially as the initial aspect ratio becomes large. This has lead Larrieu *et al.*[11] to propose an amended shallow layer framework for modelling these flows in which the initial collapse, when the grains predominantly fall vertically, is mathematically represented as a spatially distributed source of mass to the flowing layer. With this model, Larrieu *et al.*[11] were able to reconcile some of the differences between the theoretical and experimental observations, though to obtain quantitative agreement required the use of an unrealistically large coefficient for the basal friction law. Denlinger and Iverson[12] adopted a different strategy for including nonhydrostatic effects, formulated for flows over irregular topography: essentially vertical accelerations are retained, which when of significant magnitude lead to the reduction of the basal normal stress. This approach, coupled to a Coulomb-type constitutive law, permitted the computation of the flow over irregular surfaces for which it was cumbersome to form a curvilinear coordinate in which one of the axes is always perpendicular to the underlying bed.[12] Another assumption inherent to the use of shallow, single-flowing-layer models is that all of the grains throughout the layer are in motion and their velocity is adequately represented by a depth-averaged value. Lajeunessse *et al.*[4] and Lube *et al.*[13] have shown experimentally that there is a growing, static, basal layer over which grains flow and this process may not be well captured in simple depth-averaged models of the motion.

In this paper we study experimentally and theoretically two-dimensional granular slumps down slopes. Grains are released from rest behind a barrier that is rapidly removed, flow down an inclined plane and are eventually arrested because the inclination is always less than the angle of friction between the boundary and the mobile particles. The deposit formed by these flows is considerably extended relative to those over horizontal surfaces due to the downslope acceleration. The flows themselves are relatively thin: for example, after they have arrested a typical ratio of their length to depth is 100. The proportion of their motion during which there is significant acceleration perpendicular to the plane is small. Thus we propose that a shallow-layer model of the flow is appropriate for a large portion of the motion and we employ the model formulated by Refs. 9 and 10, noting that Kerswell[9] has presented quasianalytical and numerical results for dam-break flows down slopes, resisted by a Coulomb drag. In this contribution, we establish a regime in which the particles propagate a long distance from their source and we develop an asymptotic analysis to reveal how the maximum height of the arrested particles, h_∞, depends upon the aspect ratio. We demonstrate that this regime corresponds to a scenario in which the inclination of the plane is relatively close to the angle of friction, or, for flows over a horizontal plane, when the initial aspect ratio is large. Thus this analysis will also illuminate the scaling relationship found numerically by Kerswell[9] and experimentally by Lajeunesse *et al.*[4] and Balmforth and Kerswell.[3]

This paper is structured as follows: First we formulate the problem using a shallow layer model and develop the asymptotic analysis (Sec. II). We then report our experimental measurements in Sec. III and compare it to the theoretical predictions. Finally in Sec. IV, we summarize our findings and indicate what this study reveals about the modelling of granular materials. We also note that the scaling analysis that underlies the asymptotic expansion may be applied to axisymmetric flows over horizontal surfaces and we demonstrate in Sec. II C that this reveals a relationship between h_∞/h_0 and the initial aspect ratio borne out through numerical computations and some experimental studies.

II. SHALLOW-LAYER MODELS AND ANALYSIS

A. Formulation

There is a long tradition of using shallow-layer models to capture the essential physics of hydraulic and other naturally occurring particulate and granular flows (e.g., Ref. 14). These models are based upon the flows being predominantly parallel to the underlying boundary so that accelerations perpendicular to the boundary are negligible and the pressure is essentially hydrostatic. Such descriptions have also been applied to dam-break flows, where the fluid is instantaneously set into motion by the removal of a dam and aside from the very initial phases of the motion that include nonhydrostatic effects, the shallow-water equations have been shown to represent the motion accurately (see Hogg and Pritchard,[15] and references therein, for a discussion of these flows). For granular flows there have also been several recent studies in which the flowing grains are modelled as a continuum, using a shallow-layer description.[16–19] These studies have demonstrated that the flowing granular layer exhibits several phenomena usually associated with the motion of fluids, such as normal and oblique shocks, and that these can be accurately predicted by the shallow-layer models.[18,19]

In this study we employ a shallow layer model for unsteady granular collapses down inclined planes, essentially identical to the formulations of Mangeney-Castelnau *et al.*[20] and Kerswell.[9] We treat the grains as a flowing continuum with a constant bulk density, a layer depth $h(x,t)$ and a velocity field $u(x,z,t)$, where the x and z coordinate axes are orientated parallel and perpendicular to the inclined plane. The depth-averaged expression of conservation of mass per unit width is then given by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h u\,\mathrm{d}z\right) = 0. \tag{2.1}$$

Since the flow is predominantly parallel to the underlying boundary, which is inclined at angle θ to the horizontal, the leading order, normal component of the pressure tensor is given by

$$p_{zz} = \rho g \cos\theta (h - z), \tag{2.2}$$

where ρ is the bulk density. The streamwise component, p_{xx} may be related to p_{zz} by

$$p_{xx} = K p_{zz}, \tag{2.3}$$

where for deformable Coulomb-type materials Savage and Hutter,[16] showed that the constant of proportionality, K, is related to the internal and basal angles of frictions, while for more rapidly flowing scenarios Pouliquen and Forterre,[17] Gray *et al.*,[18] Hákonardóttir and Hogg,[19] and Jop *et al.*[21] have suggested $K=1$, corresponding to an isotropic pressure field. Then the depth-averaged downslope momentum equation yields

$$\frac{\partial}{\partial t}\left(\int_0^h \rho u\,\mathrm{d}z\right) + \frac{\partial}{\partial x}\left(\int_0^h \rho u^2 + p_{xx}\,\mathrm{d}z\right) = \rho g \sin\theta h - \tau_b, \tag{2.4}$$

where τ_b is the basal shear stress exerted by the flow. To close this system of equation we write

$$\int_0^h u\,\mathrm{d}z = \bar{u}h \quad \text{and} \quad \int_0^h u^2\,\mathrm{d}z = \bar{u}^2 h, \tag{2.5}$$

which is equivalent to assuming a plug-like velocity profile. While we anticipate that there may be shear in the velocity profile, as has been shown for flows over horizontal surfaces,[4,13] we comment that this modelling assumption is common even for flows of fluid for which the basal condition of no-slip introduces significant shear into the velocity profile. [Hogg and Pritchard[15] analyze situations in which (2.5) is relaxed to reflect the presence of velocity shear and demonstrate that the predictions from shallow-layer models may be strongly affected.] We must also prescribe a model for the boundary shear stress, τ_b. Here there have been many proposed formulations, including Coulomb-type models of sliding friction[16] and velocity dependent friction angles,[21] as well as other empirical drag-laws that relate the basal resistance to the relative velocity between the flowing layer and the boundary. In this study we adopt a simple description, namely, a Coulomb-type drag with a constant angle of friction between the grains and the boundary that can be experimentally measured (see Sec. III). We note that this model has some failings: it cannot predict fully developed flow with uniform velocities and depths unless the angle of inclination of the plane exactly matches the angle of friction, but in this study we are interested in flows that arrest for which we postulate that a Coulomb-type drag is the dominant force. Thus we write

$$\tau_b = \rho g \cos\theta h \tan\delta\,\mathrm{sgn}(u), \tag{2.6}$$

where the coefficient of friction is $\tan\delta$, δ denotes the angle of basal friction between the grains and the underlying plane and $\mathrm{sgn}(u)=u/|u|$. This model is to be applied to flows that accelerate, flow down the plane, and are eventually arrested. Hence the frictional drag eventually exceeds that arising from the downslope component of gravitational acceleration and from the streamwise gradient of the pressure tensor.

The flows are initiated from dam-break initial conditions,

$$\bar{u} = 0, \quad h = h_0 \quad \text{for} \quad 0 \le x \le x_0. \tag{2.7}$$

The abrupt change of height that occurs when the dam is removed means that accelerations normal to the boundary are initially unlikely to be small; Lube *et al.*[1] indicate that for flows over horizontal surfaces approximately half of the duration is spent with significant vertical motion. For granular slumps down slopes, however, the initial nonhydrostatic period is likely to be much less important because the downslope motion is considerably extended and thus most of the motion occurs as a thin layer flowing over the underlying boundary. The front of the motion, $x_f(t)$, is given by the first downstream position at which $h(x_f,t)=0$.

We now introduce dimensionless variables, by scaling the downslope distance, x and layer height $h(x,t)$ by x_0 and h_0, respectively. Downslope velocity is rendered dimensionless by $(Kg\cos\theta h_0)^{1/2}$ and the time by $x_0/(Kg\cos\theta h_0)^{1/2}$. Then replacing all variables by their dimensionless counterparts and denoting the depth-averaged velocity by $u(x,t)$, we obtain the following governing equations (cf. Ref. 9):

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \tag{2.8}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = -\frac{x_0}{Kh_0}(\tan\delta - \tan\theta)\mathrm{sgn}(u). \tag{2.9}$$

The initial conditions are given by $u=0$ and $h=1$ for $0 \le x \le 1$. Boundary conditions are that the velocity vanishes at the back wall ($u(0,t)=0$) and that the front is advected kinematically ($\mathrm{d}x_f/\mathrm{d}t = u(x_f,t)$). The sole dimensionless parameter is denoted by

$$\epsilon^3 = \frac{x_0}{Kh_0}(\tan\delta - \tan\theta). \tag{2.10}$$

We note that ϵ depends upon both the difference between the basal friction angle and the angle of inclination of the slope, the initial aspect ratio, $a \equiv h_0/x_0$ and the Earth pressure coefficient relating p_{xx} to p_{zz}, given by

$$K = 2\sec^2\phi\left(1 - \mathrm{sgn}\left(\frac{\partial u}{\partial x}\right)\left(1 - \frac{\cos^2\phi}{\cos^2\delta}\right)^{1/2}\right) - 1, \tag{2.11}$$

where ϕ is the angle of internal friction.[16] This coefficient varies discontinuously with $\partial u/\partial x$, but for the flows under consideration here it turns out that $\partial u/\partial x > 0$ and so the coefficient always takes its "active" value.

Kerswell[9] adopted an identical system of equations (up to the choice of dimensional scales) for the unsteady slumping of granular materials over horizontal and sloping surfaces, which he integrated using analytical and numerical techniques. Further he numerically elucidated the dependence of the maximum height of the deposit upon the initial aspect ratio in the regime $a \gg 1$ (equivalently $\epsilon \ll 1$) to show that $h_\infty \sim \epsilon^2$. In the current scenario, we note that this regime ($\epsilon \ll 1$) is accessible not only by studying flows of large initial aspect ratio, but also by studying flows on relatively steep slopes ($0 < \delta - \theta \ll 1$). Thus the regime $\epsilon \ll 1$ is relevant for moderate aspect ratio releases on steep slopes and so the use of a shallow-layer model is more likely to be appropriate because the flows rapidly become thin during their motion down the plane and the initial phase, during which the pressure is different from hydrostatic, is short relative to the total duration. This is in contrast to flows over horizontal surfaces ($\theta = 0$) for which the regime $\epsilon \ll 1$ is accessible only for large initial aspects ratios when the flows may be far from "hydrostatic" during much of their motion.

We note immediately that the front of the flow, $x_f(t)$, may be calculated exactly; treating the equations as a hyperbolic system, it may be readily shown that $\mathrm{d}/\mathrm{d}t(u + 2h^{1/2}) = -\epsilon^3$ on characteristics given by $\mathrm{d}x/\mathrm{d}t = u + h^{1/2}$. The leading characteristic corresponds to the front $(h(x_f,t)=0)$ and then using the initial conditions, $u(x_f,t) = 2 - \epsilon^3 t$. Hence the front position is given by $x_f(t) = 1 + 2t - \frac{1}{2}\epsilon^3 t^2$ and thus the maximum dimensionless runout length occurs at $t = 2/\epsilon^3$ and the runout length is given by $x_\infty = 1 + 2/\epsilon^3$ (see Ref. 15).

B. Asymptotic analysis

We now analyze the governing partial differential equations (2.8) and (2.9) in the regime $\epsilon \ll 1$ to elucidate how the final height of the deposit depend upon ϵ.

This dependence is readily seen through the following scaling analysis: initially the drag force, represented through the right-hand side of (2.9) is negligible and the granular material flows away from the dam under a dynamic balance between the streamwise pressure gradient and its inertia. The front of the material moves at constant speed, $x_f \sim t$ (see the classical dam-break solution reported by Whitham[14] among others) and thus to conserve mass $h \sim 1/t$. The drag force becomes non-negligible when the inertia ($\partial u/\partial t$), the pressure gradient ($\partial h/\partial x$), and the drag force (ϵ^3) become comparable. This demands

$$\frac{u}{t} \sim \frac{h}{x} \sim \epsilon^3. \tag{2.12}$$

For kinematic consistency $u \sim x/t$ and thus $x \sim \epsilon^3 t^2$. Furthermore, matching to the drag free solution requires $h \sim 1/t$ and thus drag becomes dynamically important and may arrest the motion when time scales $t \sim 1/\epsilon^2$, length scales $x \sim 1/\epsilon$, and the height of the flow $h \sim \epsilon^2$. This scaling is in accord with the computations of Kerswell[9] and the experimental observations of Lajeunesse *et al.*[4]

We now embody this scaling analysis into an asymptotic analysis of the governing equations, in which we match the initial drag-free behavior to the drag-affected motion at later times. The drag-free motion from a lock of finite extent is implicitly given by

$$t = \frac{64}{(2-\beta)^{3/2}(\alpha+2)^{3/2}} F\left[\frac{3}{2},\frac{3}{2};1;\frac{(2-\alpha)(\beta+2)}{(2-\beta)(\alpha+2)}\right], \tag{2.13}$$

where $\alpha = u + 2\sqrt{h}$, $\beta = u - 2\sqrt{h}$ and F denotes a hypergeometric function.[22] This expression is valid for $t > 1$ and $0 < x < 1 + 2t - 3t^{2/3}$. Thus to leading order in the regime $t \gg 1$, we find that

$$u = \frac{x}{t} + \cdots \quad \text{and} \quad h = \frac{1}{\pi t}\left(4 - \frac{x^2}{t^2}\right)^{1/2} + \cdots. \tag{2.14}$$

As demonstrated above these expression for $h(x,t)$ and $u(x,t)$ are no longer accurate representations of the motion when drag becomes non-negligible on time scales $t \sim 1/\epsilon^2$ and length scales $x \sim 1/\epsilon$ when the velocity $u \sim \epsilon$ and $h \sim \epsilon^2$. Using this distinguished scaling we introduce new independent variables $T = \epsilon^2 t$ and $X = \epsilon x$ and dependent variables $U(X,T) = u(x,t)/\epsilon$ and $H(X,T) = h(x,t)/\epsilon^2$ to find that the governing equations are now given by

$$\frac{\partial H}{\partial T} + \frac{\partial}{\partial X}(UH) = 0, \tag{2.15}$$

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + \frac{\partial H}{\partial X} = -1. \tag{2.16}$$

These equations are subject to the boundary conditions that $U(0,T) = 0$ and thus from (2.16) $\partial H/\partial X(0,T) = -1$. Also the matching conditions demand

$$U \to \frac{X}{T} \quad \text{and} \quad H \to \frac{2}{\pi T} \quad \text{as} \quad T \to 0. \tag{2.17}$$

The motion first arrests at the back wall when $\partial H/\partial T$ first vanishes at $X = 0$. From (2.15), this is equivalent to finding when $\partial U/\partial X = 0$ at $X = 0$.

We transform (2.15) and (2.16) by introducing the new independent variable $y = X/T$ and dependent variables $\hat{H} = TH$ and $Q = \hat{H}(U - y)$ to remove the singularity in the matching condition (2.17). This gives

$$\frac{\partial \hat{H}}{\partial T} = -\frac{1}{T}\frac{\partial Q}{\partial y}, \tag{2.18}$$

$$\frac{\partial Q}{\partial T} = -\frac{1}{T}\left(Q + \frac{\partial}{\partial y}\left(\frac{Q^2}{\hat{H}} + \frac{\hat{H}^2}{2T}\right) + T\hat{H}\right), \tag{2.19}$$

with initial conditions $Q(y,0) = 0$ and $\hat{H}(y,0) = 2/\pi$ and boundary conditions $Q(0,T) = 0$ and $\partial \hat{H}/\partial y(0,T) = -T^2$. These equations are integrated numerically using a finite difference scheme and a second-order Lax-Wendroff algorithm. The integration poses few numerical challenges since within this asymptotic domain close to the back wall, there is no need to resolve the motion of the front of the flow. We find that the flow arrest at the back wall at $T \equiv T_s = 1.278$ and that $\hat{H}(0,T_s) = 1.476$. Thus in terms of the original dimensionless variables we have calculated

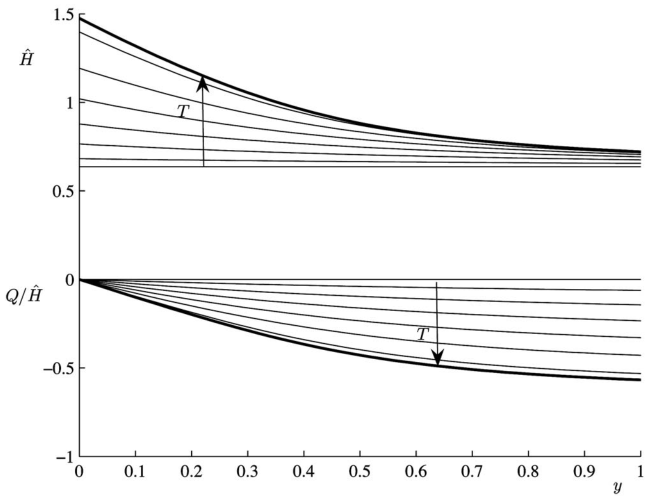

FIG. 2. The evolution of $\hat{H}(y,T)$ and $V(y,T)\equiv Q/\hat{H}$ until the flow arrests at $T\equiv T_s=1.278$. The profiles are shown at $T=0,0.2,0.4,0.6,0.8,1.0,1.2$, and 1.278. The final profiles are plotted with a thicker line.

$$t_s=\frac{1.278}{\epsilon^2} \quad \text{and} \quad h_\infty=1.155\epsilon^2. \tag{2.20}$$

We plot in Fig. 2 profiles of $\hat{H}(y,T)$ and $V(y,T)\equiv Q/\hat{H}$ showing how the flow approaches the arrested state close to the back wall. Note that the stopping condition is given by the earliest time at which $\partial V/\partial y=-1$ at $y=0$.

Kerswell[9] integrated the governing equations (2.8) and (2.9) numerically by dividing the domain into Lagrangian cells that were advected with the local velocity. The front cell, marking the foremost edge of the flowing material, was handled differently; its position was determined by the lead characteristic, which could be analytically calculated as discussed above. We may employ this Lagrangian numerical technique to this problem to analyze the form of the solution when $\epsilon \ll 1$; in this regime, many Lagrangian cells are required to resolve the motion fully and the run times are slow. However we find numerical evidence that confirms the asymptotic calculations developed above. Comparisons between h_∞ calculated asymptotically and numerically through this Lagrangian element scheme are shown in Fig. 4 below, showing reasonable agreement up to $\epsilon \approx 0.7$.

C. Axisymmetric motion over a horizontal plane

A similar scaling analysis may be employed for axisymmetric flows over a horizontal surface to reveal how the maximum depth of the deposit depends upon the initial aspect ratio. In axisymmetry geometry, the velocity and height fields are now functions of the radius r and t and in terms of dimensionless variables, conservation of mass and momentum are given by

$$\frac{\partial h}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ruh)=0 \quad \text{and} \quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{\partial h}{\partial r}=-\frac{1}{A}, \tag{2.21}$$

here $A=Kh_0/[x_0 \tan \delta]$.[9,10] We examine the dependence of the maximum height of the deposit, h_∞ upon A when $A \gg 1$. Initially the drag-free motion leads to the front moving with constant speed $r \sim t$ and so by conservation of mass, $h \sim 1/t^2$. Then the time scale on which drag becomes comparable to the inertia and pressure gradient is given by demanding $u/t \sim h/r \sim 1/A$. Kinematic consistency $(u \sim r/t)$ and matching to the drag free solution then leads to the time scales, length scales and depth of the arrested layer scaling as

$$t \sim A^{1/2} \quad r \sim 1 \quad \text{and} \quad h \sim 1/A. \tag{2.22}$$

This scaling analysis indicates that $h_\infty \sim 1/A$, which is in accord with the numerical findings of Kerswell[9] and the experimental observations of Lajeunesse *et al.*;[4] it differs slightly from the observations of Lube *et al.*[1] who find that $h_\infty \sim A^{-5/6}$ in this regime.

III. EXPERIMENTS

A. Method

A variety of dry granular materials were released down a wooden chute, inclined to the horizontal at an angle between 10°–25°. The chute was of length 3 m and width 30 cm and the particles were initially in place behind a wooden lockgate that spanned the width of the chute, was perpendicular to the base of the chute and at a distance x_0 from the back wall. Granular material was poured into the lock and the top surface was carefully smoothed to give a uniform depth of material, h_0, throughout. The lockgate was rapidly removed by hand to initiate the flow down the plane. Once the material had come to rest the resulting deposit was measured to determine the runout length, x_∞ and the distribution of the deposited material in terms of its depth, measured at various downslope locations by carefully inserting a thin metal ruler through the grains so that they were minimally disturbed. In particular the depth at the back of the lock, h_∞, was measured. The flow and the deposit exhibited some variation across the channel width, although they were predominantly two-dimensional. Measurements of the depths and the runout length were made at five positions across the chute and were averaged.

Four different materials were used in the experiments and their properties are given in Table I. The internal angle of friction, ϕ, was determined by building a symmetric cone of particles on a horizontal layer of similar particles and repeatedly measuring the height and diameter of the cone. A dynamic bed friction angle, δ, was determined by pouring a thin layer of particles down the plane and measuring the maximum angle of inclination of the plane when the flow was arrested and motion could not be sustained. Given these measurements, the active Earth pressure coefficient may be evaluated from (2.11); it lies in the range 0.81–0.91 for these four materials. A static angle of basal friction was also measured by tilting a plane on which there was a stationary, thin layer of particle and finding the smallest angle at which flow was initiated. As expected, the dynamic friction angle was less than the static friction angle for all the materials. For some of the experimental runs, the base of the chute was lined with sandpaper, roughening the slope and increasing the basal friction.

TABLE I. Material properties of the granular materials.

Material	Mean particle diameter (μm)	Bulk density (g cm^{-3})	Angle of repose ϕ (°)	Dynamic bed friction δ (°)
Ballotini	100	2.5	25.5±0.5	21±0.5
Ballotini	350	2.5	22.5±0.5	19±0.5
PVC Powder	140	0.57	19.5±0.5	25±0.5
Mustard seeds	1500	1.3	26±1	21±1

Experiments were conducted with four granular materials, various inclinations of the chute, a range of volumes of material (500–3000 cm^3) and different initial aspect ratios in the lock (h_0/x_0). Experiments, repeated from identical initial conditions, yielded similar results, providing confidence in the reproducibility of the experimental procedure. A complete description of the experimental method and results is given by Hákonardóttir.[23]

B. Results

When the chute inclination was more than a few degrees less than the basal friction, the granular materials flowed away from the lock and formed a deposit, the depth of which decreased monotonically with distance down the slope. These deposits varied approximately linearly with distance, but there remained some residual dependence on the initial aspect ratio, even if the total volume remained the same. This is illustrated in Fig. 3, where we plot the depth of the deposit scaled by the maximum depth as a function of the distance from the back wall of the lock, scaled by the runout distance for ballotini particles of mean diameter 100 μm on a 10.4° slope. Thus in accord with the theoretical developments of Sec. II, we observe that the distribution of the deposit lacks self-similarity and depends upon the initial aspect ratio of the release.

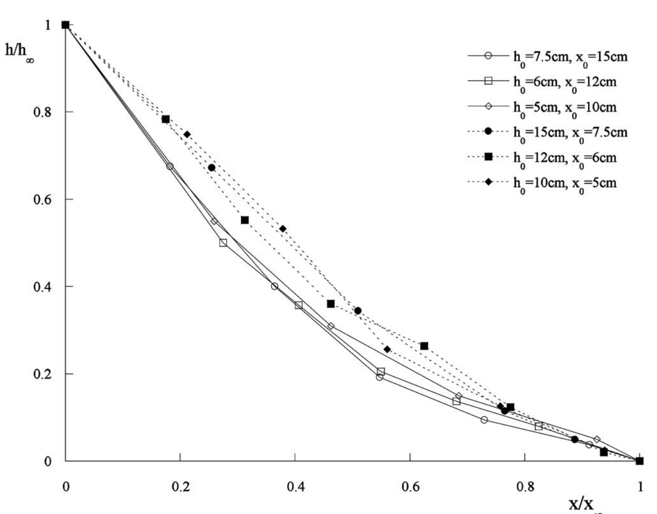

FIG. 3. The depth of the deposit scaled by the depth at the back wall, h/h_∞, as a function of the downstream distance scaled by the runout length, x/x_∞. The data series are for flows of ballotini particles of mean size 100 μm down a plane inclined at 10.4° to the horizontal. The open symbols and solid lines correspond to an initial aspect ratio, $h_0/x_0=0.5$ and the closed symbols and dashed lines to $h_0/x_0=2.0$.

We now consider the bulk descriptors of the deposits formed by these granular slumps, namely, in terms of dimensional variables, the depth at the back wall, h_∞ and the runout length, x_∞. In Fig. 4, the dimensionless depth h_∞/h_0 is plotted as a function of ϵ^2 (note that there is good agreement between the theoretical prediction and the experimental measurements). This dependence on ϵ is in agreement with the empirical forms determined for flows over horizontal surfaces by Lajeunesse *et al.*[4] and quite close to the forms proposed by Balmforth and Kerswell[3] and Lube *et al.*[5] ($h_\infty/h_0 \sim a^{-0.6} \sim \epsilon^{1.8}$). However, for these flows down slopes, we have found quantitative agreement between theory and experiments, with no adjustable fitting parameters. We note that some the data points have relatively large error bars. This arises because of the uncertainties in measuring the friction angles; for flows down relatively steep slopes, even small measurement errors in δ and ϕ lead to relatively large errors in ϵ. We also note that there is no systematic divergence from the theoretical prediction for each of the granular materials

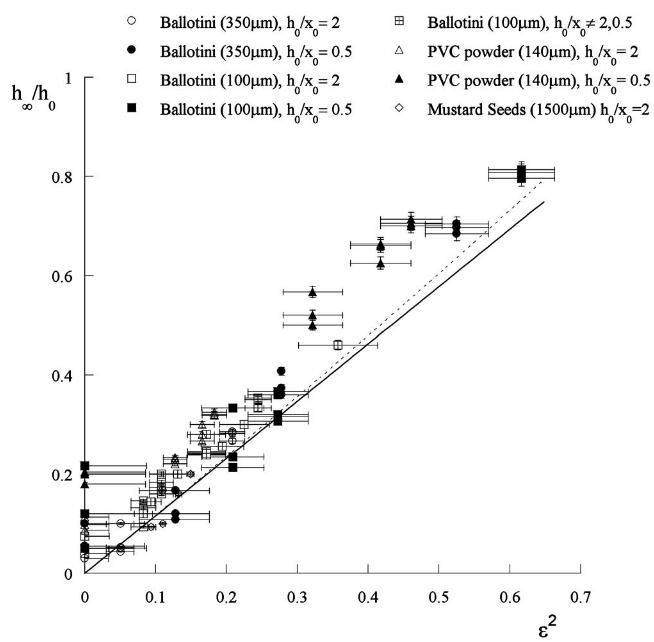

FIG. 4. The depth of the deposit at the back of the lock relative to the initial depth, h_∞/h_0, as a function of ϵ^2. The data includes experiments with particles of ballotini (100 μm and 350 μm), PVC powder and mustard seeds. The solid line (—) is the asymptotic relationship $h_\infty/h_0=1.155\epsilon^2$ (2.20) and the dashed line (- - -) arises from the numerical integration of the full governing equations.

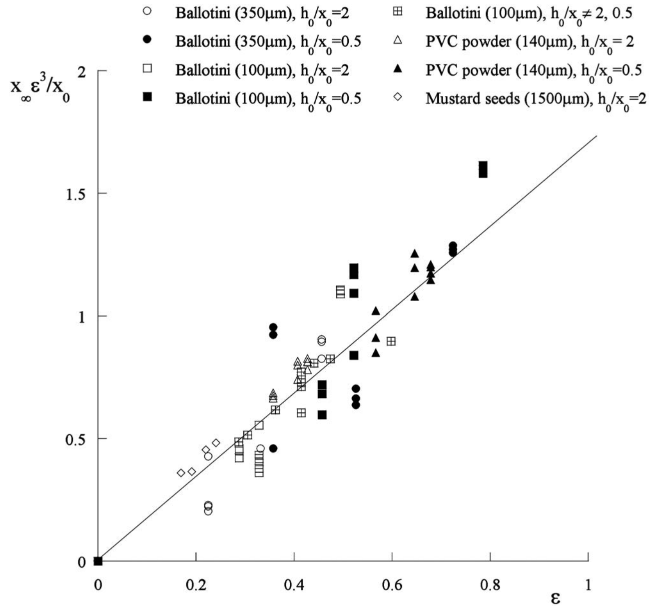

FIG. 5. The scaled runout length, $x_\infty \epsilon^3/x_0$, as a function of ϵ. The data include experiments with particles of ballotini (100 μm and 350 μm), PVC powder and mustard seeds. The solid line (——) is an estimate for x_∞ derived by assuming the depth of the deposit varies linearly with downstream distance.

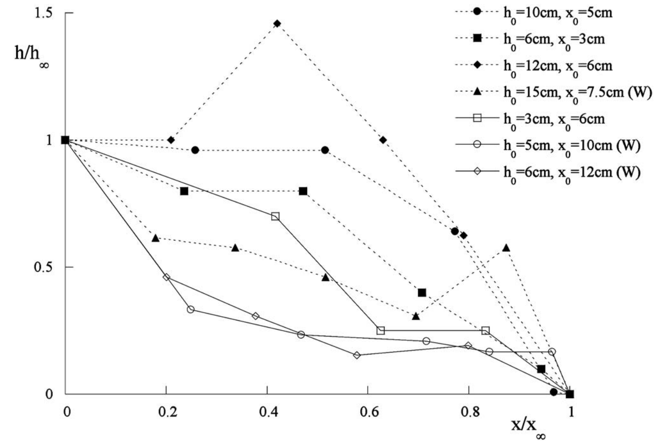

FIG. 6. The depth of the deposit scaled by the depth at the back wall, h/h_∞, as a function of the downstream distance scaled by the runout length, x/x_∞. The data series are for flows of ballotini particles of mean size 350 μm down a plane inclined at 18° to the horizontal. The open symbols and solid lines correspond to an initial aspect ratio, $h_0/x_0=2$ and the open symbols and dashed lines to $h_0/x_0=0.5$. Those data series marked with a (W) exhibited a wave-like disturbance that significantly extended the flow.

and for each of the initial aspect ratios; rather, the dependence upon the initial and material properties is appropriate encompassed within the single parameter ϵ.

Turning then to the runout length we find a difference between the theoretical prediction and the measurements. From dam-break initial conditions, the model with Coulomb drag predicts that the runout length $x_\infty \epsilon^3/x_0 = 2+\epsilon^3$. However we find empirically that $x_\infty \epsilon^3/x_0 \sim \epsilon$. This is in accord with the experimental investigations of Lajeunesse *et al.*[4] and Lube *et al.*[5] in the regime $\epsilon \ll 1$, while slightly different from Balmforth and Kerswell,[3] who suggested $x_\infty \epsilon^3/x_0 \sim \epsilon^{0.3}$. Moreover this correlation conserves mass in a "bulk" sense, because $h_\infty l_\infty \sim h_0 x_0$; but the runout is smaller than predicted by the simple Coulomb theory.

There are a number of possible explanations for this difference: one possibility is that the motion is far from "hydrostatic" and there are significant vertical accelerations, especially on initiation. This led Larrieu *et al.*[11] to propose a revised representation of the initial stages of the flow. In these downslopes slumps, the runout is considerably extended relative to those over horizontal surfaces and so the initial collapse occurs over a relatively short period of the flow and the majority of the motion occurs as a thin layer. Another possibility is that the velocity field is far from pluglike and so the depth-integrated closure (2.5) is inappropriate. Some investigations have included "shape factors" to account for shear in the velocity profile and these can strongly affect the calculate motion close to the front of dambreak flow.[15] Alternatively the basal resistance may be inappropriately represented by a Coulomb drag, with a constant angle of basal friction; certainly the flows are relatively fast and thin at the front and possibly the angle of friction should depend upon the local velocity and height, as suggested by Jop *et al.*[21] Additionally the physical processes that dominate the arresting phase of this motion may not be represented in the mathematical model. Based on experimental observations of slumps over horizontal surfaces, Mangeney-Castelnau *et al.*[10] suggest that the arrest should be treated as a vertically propagating front between mobile and static material and this has been explored experimentally by Lajeunesse *et al.*[4] and Lube *et al.*,[13] who measured the growth of the underlying basal layer. We comment that processes such as this are not represented in the current shallow layer model. We have added a trend line to Fig. 5, which was computed by assuming that the thickness of the deposit varied linearly with distance and thus $x_\infty/x_0 = 2/[1.155\epsilon^2]$. We note that this estimate, derived using mass conservation, is consistent with the experimental data.

Finally we report that when the chute inclination becomes close to the basal angle of friction a rather different deposit is formed; in some cases it no longer varies monotonically with distance and exhibits an interior maximum (see Fig. 6). This effect could be associated with the experimental method and nonsystematic variations in the removal of the lockgate. However, a transient profile of this morphology was noted in numerical simulations of Staron and Hinch,[8] who viewed the interior maximum as a "wave" carrying mass flux away from the collapse, but they did not find any final states of this nature. An additional curiosity occurred with the slumps of ballotini grains of mean diameter 350 μm, where the material flowed out to a certain distance and then the deposit "refailed" and a wave-like structure propagated down the slope, reworking the deposit and significantly extending the flow. We postulate that this phenomenon is due to localized slipping, generating a disturbance that propagated through the previously stationary grains. We note that in terms of the theoretical description developed in Sec. II, "wave-like" motion is supported by the hyperbolic governing equations and could be readily generated by a localized disturbance. We speculate that the origin of this slippage is that the bed friction is not adequately represented by a constant value for these granular media, which have some polydispersity and exhibit variations in shape. These lead to minor variations in the magnitude of the bed friction, which could generate localized flows when the inclination of the plane is relatively steep. For slumps that exhibit a nonmonotonic profile and for those affected by a wave-like disturbance, however, we note that the measured values of the depth of the deposit at the back wall are included in Fig. 4 and are broadly in line with the theoretical predictions. This suggests that even though complex flow patterns may occur in the interior of these slumps, the slow motion at the rear of the flow is resisted by a Coulomb-type drag.

IV. CONCLUSION

Slumps of granular materials down slopes provide an interesting and challenging setting in which to study flow and arrest of particulate systems. Downslope acceleration significantly extends these motions relative to their counterparts over horizontal surfaces and so much the flow occurs as a thin layer, which means that mathematical models based on this shallowness are more likely to represent the dynamics accurately. In this paper we have employed a simple dynamical model by assuming the basal drag may be represented through a constant friction coefficient. From dam-break initial conditions, this implies that the motion is characterized by a single dimensionless parameter ϵ, defined by (2.10), and that ϵ is much smaller than unity not only when the aspect ratio of the initial release is large, but also when the inclination of the plane is close to the angle of basal friction. Thus in contrast to studies of flow over horizontal surfaces, we may readily access the regime $\epsilon \ll 1$ without resorting to flows of large aspect ratio for which the "nonshallow" initial collapse appears to influence the motion very strongly.

The regime $\epsilon \ll 1$ also corresponds to a weak resistive force, and this forms the basis for an asymptotic analysis of the governing equations that reveals the dependence of the final height of the deposit, h_∞ upon ϵ. Essentially at early times the flows slump, driven by the pressure gradient, and drag is negligible, while at later times drag becomes dynamically important and eventually arrests the flow. These processes may be embodied in a matched asymptotic expansion that reveals clearly the correlation between h_∞ and ϵ, previously known only from numerical experimentation. However, this analysis also yields an analytical estimate for the maximum depth of deposit that was tested by experimentation. Our experimental results using four different granular materials agree quantitatively with the analysis. There are no fitting parameters; everything is measured independently and this is the first time slumping experiments have been modelled accurately.

Our experimental measurements for runout length, however, are not in accord with this Coulomb model. Instead they exhibit a dependence that can be rationalized using simple principles of mass conservation. These measurements share this feature with the studies of slumps over horizontal surfaces.[3–5]

This study reveals a number of features of the granular slumping problem that has been the focus for so much activity during recent years. First, because the flows down the slope are extended and thin, shallow layer models are more applicable for this scenario than for flows over horizontal surfaces and capture accurately many aspects of the unsteady motion. Indeed some of the complex flow patterns and regimes of mobile and static grains identified by the study of flows over horizontal surfaces do not carry over to these flows down slopes, where aside from the instances close to initiation, the granular material is mobilized throughout the entire flow depth. We have also demonstrated that a Coulomb drag law provides an accurate quantitative estimate of the depth of the deposit at the back wall of the lock when the flow is arrested. This analysis would not be changed significantly using the flow rule proposed by Jop *et al.*,[21] for which in the context of shallow flows, the coefficient of friction depends upon $I=ud/[gh^3]^{1/2}$, where d is the diameter of the particle. We anticipate that it should still be possible to match the initial drag-free asymptotic regime to one in which the flow is arrested. It is intriguing that this study and the previous ones over horizontal surfaces have found that the runout length is not well predicted by a Coulomb resistive force. This implies that the process of the arrest of material that was flowing rapidly is not appropriately modelled. Indeed it suggests that attention should be applied to the transition from the mobile to the arrested state. It is intriguing nevertheless that the measurements of runout from all of the two-dimensional slumping experiments show approximately the same dependence on ϵ.

Finally, we have found some curious profiles in the deposit where the depth varied nonmonotonically with distance. A possible interpretation is that the flow arrested before the material from the lock could be fully advected from the lock. Also we found that the arrested deposit was prone to localized slippage when the slope was relatively steep. This is perhaps a timely reminder that usually granular materials are not uniform and have a range of shapes, sizes and material properties.

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