UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ADVANCED FLUID DYNAMICS

MATH M0600

(Paper Code MATH-M0600)

April 2012, 2 hours 30 minutes

This paper contains **five** *questions*
A candidate's **FOUR** *best answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Candidates may use their lecture notes during the examination.

Do not turn over until instructed.

1. A layer of viscous fluid of kinematic viscosity ν and constant density ρ flows steadily down an impermeable, rigid plane inclined at an angle θ to the horizontal, driven by gravitational acceleration g. The layer is of depth h_1, measured perpendicularly to the plane.

   (a) **(7 marks)** Find the velocity and pressure fields for this flow and show that the volume flux of fluid per unit cross-slope width is $q = \frac{1}{3} g h_1^3 \sin\theta/\nu$.

   (b) **(5 marks)** Show that the rate of dissipation of energy per unit width per unit streamwise length is given by $\rho g \sin\theta q$.

   Now suppose that a second layer of fluid of thickness h_2, kinematic viscosity $\beta\nu$ and density ρ flows steadily on top of the first layer.

   (c) **(13 marks)** Determine the velocity fields in each layer and show that the total volume flux of fluid per unit cross-slope width flowing down the plane, Q, is given by

$$Q = \frac{g \sin\theta h_1^3}{3\nu} \left(1 + 3\frac{h_2}{h_1} + 3\left(\frac{h_2}{h_1}\right)^2 + \frac{1}{\beta}\left(\frac{h_2}{h_1}\right)^3 \right).$$

   Explain why the velocity field in the lower layer is independent of β.

2. (a) **(6 marks)** On the assumption that fluid inertia is negligible, show that the velocity field of an incompressible viscous fluid must satisfy

$$\nabla^2 \nabla \wedge \mathbf{u} = 0 \qquad \text{and} \qquad \nabla . \mathbf{u} = 0.$$

Then introduce a streamfunction for the two-dimensional flow, ψ, and deduce that it must satisfy $\nabla^4 \psi = 0$.

Now consider incompressible viscous fluid of kinematic viscosity ν that resides between two infinite rigid plates, which are hinged at the origin. The plates move apart from each other with constant angular velocities, $\pm\Omega$. Adopting polar coordinates (r, θ) with origin at the hinge, this means that the plates are located at $\theta = \pm\Omega t$.

(b) **(3 marks)** Write down the boundary conditions on the velocity field at each of the plates.

(c) **(10 marks)** On the assumption that fluid inertia is negligible and by seeking a solution of the form $\psi = r^2 f(\theta)$, show that the streamfunction is given by

$$\psi = -\frac{\Omega r^2 \left(\sin 2\theta - 2\theta \cos 2\Omega t\right)}{2(\sin 2\Omega t - 2\Omega t \cos 2\Omega t)}.$$

Sketch the streamlines at $\Omega t = \pi/4$.

(d) **(6 marks)** Estimate the radial distance from the hinge at which fluid inertia becomes non-negligible.

[Hint: You are given that in polar coordinates $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}^2 \frac{\partial^2}{\partial \theta^2}$*. Also the velocity field in polar coordinates,* (u_r, u_θ)*, is related to the streamfunction* $\psi(r, \theta)$ *by* $u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$ *and* $u_\theta = \frac{\partial \psi}{\partial r}$*.]*

3. A cylinder of radius a translates steadily at speed V in a direction perpendicular to its axis and parallel to a planar boundary. The centre of the cylinder remains at a distance $a + b$ from the plane and the motion takes place through incompressible fluid of dynamic viscosity μ and density ρ. The motion is to be analysed in the regime $b \ll a$.

   (a) **(5 marks)** Show that the distance between the surface of the cylinder and the plane is given approximately by

   $$h = b + \frac{x^2}{2a},$$

   where x is the distance along the planar boundary from the point of minimal separation between cylinder and plane.

   (b) **(10 marks)** Calculate the velocity field in the space between the cylinder and the plane and using the condition that the pressure is uniform sufficiently distant from the cylinder, deduce that the pressure gradient is given by

   $$\frac{\partial p}{\partial x} = \frac{\mu V}{h^2}\left(8\frac{b}{h} - 6\right).$$

   [Hint: It is convenient to work in a frame of reference with the cylinder stationary.]

   (c) **(10 marks)** Show that the magnitude of the total tangential force exerted on the plane per unit width, F, is given by

   $$F = \frac{2\pi\mu V}{b}\sqrt{2ab},$$

   and hence deduce the force that must be applied to the cylinder to maintain the motion.

[Hint: You are given that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{(m+1)}}\,\mathrm{d}x = \frac{(2m)!\pi}{2^{2m}(m!)^2}$ *for positive integers* $m \geq 0$*.]*

4. Incompressible fluid of uniform density and kinematic viscosity ν is extruded from a source at the origin and spreads predominantly radially, generating an axisymmetric motion with radial velocity u and axial velocity w ($u \gg w$). The spreading layer of thickness, $\delta(r)$, is thin relative to the radial distance, r, ($\delta \ll r$) and the radial velocity vanishes far from the plane $z = 0$. There is no externally imposed pressure gradient and the leading order terms in the radial component of the Navier-Stokes equation are given by

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

while mass conservation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0.$$

(a) **(6 marks)** Show that the quantity $\mathcal{F} = \int_{-\infty}^{\infty} u^2 r \, \mathrm{d}z$, is independent of the radial distance r.

The axisymmetric motion of this radial jet is analysed by introducing a Stokes streamfunction for the velocity field, such that

$$u = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \qquad \text{and} \qquad w = \frac{1}{r}\frac{\partial \Psi}{\partial r}.$$

(b) **(8 marks)** By balancing terms in (1), or otherwise, deduce that the thickness and streamfunction may be written as

$$\delta = \left(\frac{\nu^2}{\mathcal{F}}\right)^{1/3} r \qquad \text{and} \qquad \Psi = -\left(\mathcal{F}\nu\right)^{1/3} r f(\eta), \qquad \text{where} \qquad \eta = \frac{z}{\delta(r)}$$

and where f is a dimensionless function. Identify the Reynolds number for the motion and explain for what regime of Reynolds number the flow is relatively thin.

(c) **(4 marks)** You are given that f satisfies

$$f''' + ff'' + f'^2 = 0. \tag{2}$$

Explain why $\int_{-\infty}^{\infty} f'^2 \, \mathrm{d}\eta = 1$ and identify two other boundary conditions for f.

(d) **(7 marks)** Integrate (2) to show that

$$f = \left(\frac{3}{2}\right)^{1/3} \tanh\left[\frac{\eta}{2}\left(\frac{3}{2}\right)^{1/3}\right].$$

Determine rw as $|z/\delta| \to \infty$. Why is this quantity non-zero?

5. Inviscid fluid of uniform density flows along a rigid-walled channel ($0 \leq z \leq d$) with velocity $\mathbf{u} = U_0(z)\hat{\mathbf{x}}$ and pressure p_0, where $\hat{\mathbf{x}}$ is a unit vector along the axis of the channel. This flow is perturbed with a two-dimensional velocity field $\epsilon\mathbf{u}_1$ ($\epsilon \ll 1$), where $\mathbf{u}_1 = (u_1, 0, w_1)$ and a pressure ϵp_1.

   (a) **(6 marks)** Show that the linearised equation governing w_1 is given by

   $$\left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\nabla^2 w_1 - \frac{\mathrm{d}^2 U_0}{\mathrm{d}z^2}\frac{\partial w_1}{\partial x} = 0.$$

   (b) **(4 marks)** Introduce $w_1 = (U_0 - c)\phi(z)\exp(\mathrm{i}k(x - ct))$ and show that

   $$\frac{\mathrm{d}}{\mathrm{d}z}\left((U_0 - c)^2\frac{\mathrm{d}\phi}{\mathrm{d}z}\right) - k^2(U_0 - c)^2\phi = 0.$$

   (c) **(6 marks)** Writing $c = c_r + \mathrm{i}c_i$, deduce that

   $$\int_0^d \left((U_0 - c_r)^2 - c_i^2\right) Q \,\mathrm{d}z = 0 \qquad \text{and} \qquad c_i\int_0^d (U_0 - c_r)\, Q \,\mathrm{d}z = 0,$$

   where $Q = k^2|\phi|^2 + \left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right|^2$.

   (d) **(9 marks)** Denoting $\max(U) = U_1$ and $\min(U) = U_2$, explain why

   $$\int_0^d (U_0 - U_1)(U_0 - U_2)Q \,\mathrm{d}z < 0.$$

   Then when the flow is unstable ($c_i > 0$), show that

   $$\int_0^d (c_r^2 - c_r(U_1 + U_2) + c_i^2 + U_1U_2)Q \,\mathrm{d}z < 0,$$

   and deduce that

   $$\left(c_r - \frac{U_1 + U_2}{2}\right)^2 + c_i^2 \leq \left(\frac{U_1 - U_2}{2}\right)^2.$$

End of examination.