UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ADVANCED FLUID DYNAMICS

MATH M0600

(Paper Code MATH-M0600)

April 2013, 3 hours

This paper contains **five** *questions*
A candidate's **FOUR** *best answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Candidates may bring into the examination their lecture notes and material distributed during the course (printed or handwritten), but may not bring in textbooks.

In cylindrical polar coordinates (r, θ, z), with velocity field $\mathbf{u} = (u_r, u_\theta, u_z)$, the incompressible Navier-Stokes equations with no body force are given by

$$\frac{\partial u_r}{\partial t} + \mathbf{u}.\,\nabla u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$
$$\frac{\partial u_\theta}{\partial t} + \mathbf{u}.\,\nabla u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right),$$
$$\frac{\partial u_z}{\partial t} + \mathbf{u}.\,\nabla u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z,$$

where $\nabla^2 F = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial F}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{\partial^2 F}{\partial z^2}$ and the incompressibility condition is given by

$$\nabla.\,\mathbf{u} \equiv \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

Do not turn over until instructed.

1. Viscous fluid of dynamic viscosity μ and constant density ρ occupies the semi-infinite domain above a rigid plane ($z = 0$). The plane oscillates with velocity $U\cos(\omega t)\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ denotes a unit vector parallel with the plane. The pressure in the fluid is constant and the fluid is motionless far from the plane.

   (a) **(6 marks)** Find the velocity field within the fluid and show that the shear stress at the boundary $z = 0$ is given by

   $$\tau_{xz} = -\mu U \lambda \cos(\omega t + \pi/4), \tag{1}$$

   where λ is a constant that should be explicitly be determined.

   (b) **(5 marks)** Show that the average rate of dissipation is $\dfrac{1}{2\sqrt{2}}\mu U^2 \lambda$.

   Now suppose that the fluid is bounded by another rigid plane, located at $z = d$. This plane is also forced to oscillate with velocity given by $U\cos(\omega t)\hat{\mathbf{x}}$.

   (c) **(10 marks)** Show that the fluid velocity is given by

   $$\frac{u}{U} = \mathrm{Re}\left\{ \frac{\cosh\left(\frac{1}{2}\lambda \mathrm{e}^{\mathrm{i}\pi/4}(d-2z)\right)}{\cosh(\frac{1}{2}\lambda \mathrm{e}^{\mathrm{i}\pi/4} d)} \mathrm{e}^{\mathrm{i}\omega t} \right\}.$$

   Show that the centreline velocity ($z = d/2$) vanishes when $\lambda d = 2\sqrt{2}(\frac{1}{2} + n\pi)$, where $n \in \mathbb{Z}$.

   (d) **(4 marks)** Find the shear stress at the lower boundary ($z = 0$) and evaluate it in the regimes $\lambda d \ll 1$ and $\lambda d \gg 1$.

2. (a) **(6 marks)** Show that a linear shear flow $\mathbf{u} = \Gamma z \hat{\mathbf{x}}$ may be written as

$$u_i = E_{ij} x_j + (\mathbf{x} \wedge \mathbf{\Lambda})_i \,,$$

where E_{ij} is a symmetric tensor and Λ_i is a vector, which should be explicitly determined.

A stationary spherical particle of radius a resides at the origin in a linear shear flow such that the velocity field sufficiently distant from the particle is $\mathbf{u} = \Gamma z \hat{\mathbf{x}}$. The velocity field satisfies the Stokes equations, no-slip boundary conditions on the surface of the sphere and tends to the linear shear flow far from the particle. The velocity field and the associated pressure field are given by

$$u_i = (\mathbf{x} \wedge \mathbf{\Lambda})_i \left(1 - \frac{a^3}{r^3}\right) + E_{ij} x_j \left(1 - \frac{a^5}{r^5}\right) - \frac{5 x_k E_{kl} x_l}{2r^2} \left(\frac{a^3}{r^3} - \frac{a^5}{r^5}\right) x_i,$$

$$p = -\frac{5\mu}{a^2} x_k E_{kl} x_l,$$

where Λ_i and E_{ij} are the vector and tensor determined in part (a).

(b) **(9 marks)** Show that the velocity gradient tensor, evaluated on the surface of the spherical particle is given by

$$\nabla_j u_i = \frac{3x_j}{a^2} \epsilon_{ikl} x_k \Lambda_l + \frac{5}{a^2} E_{ik} x_k x_j - \frac{5}{a^4} E_{kl} x_k x_l x_i x_j.$$

Hence show that the shear stress on the surface of the particle, τ_i, is given by

$$\tau_i = \frac{3\mu}{a} (\mathbf{x} \wedge \mathbf{\Lambda})_i + \frac{5\mu}{a} E_{ik} x_k.$$

(c) **(5 marks)** Show that the net drag on the particle vanishes.

(d) **(5 marks)** Evaluate the net torque, $\mathbf{G}$, exerted on the particle, given by

$$\mathbf{G} = \int_{r=a} \mathbf{x} \wedge \boldsymbol{\tau} \, \mathrm{d}S.$$

Hint: $\displaystyle\int_{r=a} x_i x_j \, \mathrm{d}S = \frac{4\pi a^4}{3} \delta_{ij}.$

3. A thin, two-dimensional layer of viscous fluid of height $h(x,t)$, kinematic viscosity, ν, and density, ρ, spreads gravitationally on a rigid horizontal plane. The motion is to be analysed in the lubrication regime.

   (a) **(6 marks)** Explain why the pressure adopts a hydrostatic distribution to leading order, given by

   $$p = \rho g(h - z) + p_{atm},$$

   where p_{atm} and g denote atmospheric pressure and gravitational acceleration, respectively. In this regime, what are the leading order terms in the horizontal component of the Navier-Stokes equations?

   (b) **(7 marks)** Derive the equation governing the evolution of the layer

   $$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

   (c) **(12 marks)** Now suppose the layer is of uniform thickness, h_0, apart from a small perturbation, $a(x,t)$, which is initially of the form $a(x,0) = a_0\exp(-x^2)$ $(a_0/h_0 \ll 1)$. Derive the linearised equation governing the subsequent evolution of $a(x,t)$.
   Show that the solution for $a(x,t)$ is given by

   $$a(x,t) = \frac{a_0}{f(t)}\exp\left(-\frac{x^2}{g(t)}\right),$$

   where $f(t)$ and $g(t)$ are functions to be explicitly determined.

4. Incompressible fluid of kinematic viscosity, ν, flows laminarly at high Reynolds number between two plane walls located at $y = 0$ and $y = x \tan \alpha$. Fluid is extracted at the origin where the two wall meet at a constant rate per unit width, αQ.

   (a) **(6 marks)** Show that the inviscid flow equations have solution

$$\mathbf{u} = -\frac{Q}{r}\hat{\mathbf{r}},$$

   where r is the radial distance from the origin and $\hat{\mathbf{r}}$ is a unit radial vector. Find the associated pressure distribution, $p(r)$.

   (b) **(3 marks)** Explain why this inviscid solution can not be the complete solution for the laminar flow. Determine an expression for the Reynolds number in this context.

   Now consider the boundary layer close to the planar boundary at $y = 0$ and introduce a stream function, $\psi(x, y)$, for the motion.

   (c) **(6 marks)** Show that the boundary layer equations are

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\frac{Q^2}{x^3} + \nu \frac{\partial^3 \psi}{\partial y^3}. \qquad (2)$$

   (d) **(7 marks)** Look for a similarity solution of (2) of the form

$$\psi = -\sqrt{\nu Q}\, f(\eta), \qquad \text{where} \qquad \eta = \frac{y}{x}\sqrt{\frac{Q}{\nu}}$$

   and show that the governing equation for f is given by

$$f''' - f'^2 + 1 = 0.$$

   (e) **(3 marks)** What are the boundary conditions satisfied by f and its derivatives?

5. An incompressible, inviscid fluid flows between two concentric, infinite cylinders of radii R_1 and R_2 ($R_2 > R_1$) with velocity field $U_\theta(r)\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ denotes a unit vector in the angular direction and r is the radial distance from the axes of the cylinders. The motion has a corresponding pressure field, $P(r)$. The velocity and pressure fields are perturbed by a small disturbance so that

$$\mathbf{u} = U_\theta(r)\hat{\boldsymbol{\theta}} + \mathbf{u}' \qquad \text{and} \qquad p = P(r) + p',$$

where the time-dependent disturbance fields are assumed to be functions of only r, z and t. Here z is the axial coordinate.

(a) **(8 marks)** Write down the linearised Euler equations and incompressibility condition in terms of $\mathbf{u}' = (u_r', u_\theta', u_z')$ and p'.

(b) **(9 marks)** On the assumption that $\mathbf{u}' = \mathrm{e}^{\mathrm{i}(kz+\omega t)}\left(\hat{u}_r(r), \hat{u}_\theta(r), \hat{u}_z(r)\right)$ and $p' = \hat{p}(r)\mathrm{e}^{\mathrm{i}(kz+\omega t)}$, derive

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}r}\right) = \left(k^2 + \frac{1}{r^2} - \frac{k^2}{r^3\omega^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(U_\theta^2 r^2\right)\right) r\hat{u}_r.$$

(c) **(8 marks)** Show further that

$$k^2\int_{R_1}^{R_2}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(U_\theta^2 r^2\right)\hat{u}_r^2\,\mathrm{d}r = \omega^2\int_{R_1}^{R_2}\Phi\,\mathrm{d}r,$$

where Φ is to be determined. Hence deduce that if the flow is linearly unstable then $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(U_\theta^2 r^2\right) < 0$ at some value of r ($R_1 < r < R_2$).

End of examination.