UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ADVANCED FLUID DYNAMICS

MATH M0600

(Paper Code MATH-M0600)

January 2014, 3 hours

This paper contains **five** *questions*
A candidate's **FOUR** *best answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Candidates may bring into the examination their lecture notes and material distributed during the course (printed or handwritten), but may not bring in textbooks.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Incompressible fluid of kinematic viscosity ν and density ρ flows between two infinite stationary parallel plates, located at $z = 0$ and $z = h$. The motion is driven by a pressure gradient given by

$$\frac{\partial p}{\partial x} = -\rho G \cos 2\omega t.$$

   (a) **(4 marks)** Simplify the Navier-Stokes equations to derive a governing equation for the velocity field $\mathbf{u} = u\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a unit vector along the x-axis. What are the boundary conditions?

   (b) **(12 marks)** Find the flow field.
   [Hint: you may wish to write $u = Re\{f(z)e^{2i\omega t}\}$, where $Re\{...\}$ denotes the real part.]

   (c) **(3 marks)** Evaluate the velocity averaged across the width of the channel.

   (d) **(6 marks)** Show that the average flow velocity in the regime $h\delta \ll 1$, where $\delta = \sqrt{\omega/\nu}$, is given by

$$\frac{Gh^2}{12\nu} \cos 2\omega t.$$

   Explain how this result could have been anticipated.

2. (a) **(5 marks)** Sketch the streamlines of the linear straining flow $u_i = E_{ij}x_j$, where

$$E_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

A stationary spherical particle of radius a resides at the origin of a linear straining flow such that the velocity field sufficiently distant from the particle is $u_i = E_{ij}x_j$. The velocity field satisfies the Stokes equations, no-slip boundary conditions on the surface of the sphere and tends to the linear straining flow far from the particle. The velocity field and the associated pressure field are given by

$$u_i = E_{ij}x_j\left(1 - \frac{a^5}{r^5}\right) - \frac{5x_k E_{kl} x_l}{2r^2}\left(\frac{a^3}{r^3} - \frac{a^5}{r^5}\right)x_i,$$

$$p = -\frac{5\mu}{a^2}x_k E_{kl} x_l.$$

(b) **(10 marks)** Show that the velocity gradient tensor, evaluated on the surface of the spherical particle is given by

$$\nabla_j u_i = \frac{5}{a^2}E_{ik}x_k x_j - \frac{5}{a^4}E_{kl}x_k x_l x_i x_j.$$

Hence show that the shear stress on the surface of the particle, τ_i, is given by

$$\tau_i = \frac{5\mu}{a}E_{ik}x_k.$$

(c) **(5 marks)** Show that the net drag on the particle vanishes.

(d) **(5 marks)** Evaluate the net torque, $\mathbf{G}$, exerted on the particle, given by

$$\mathbf{G} = \int_{r=a} \mathbf{x} \wedge \boldsymbol{\tau} \, \mathrm{d}S.$$

Hint: $\displaystyle\int_{r=a} x_i x_j \, \mathrm{d}S = \frac{4\pi a^4}{3}\delta_{ij}.$

3. A circular disk hovers on a cushion of air above an 'air' table, which is a horizontal, porous surface through which air is pumped at a constant rate. The disk is of radius R and weight Mg and it is in equilibrium at a distance h above the air-table. The air, which has dynamic viscosity μ and density ρ, is pumped so that it emerges through the porous table with a volume flux V per unit area of the table surface.

   The motion is to be considered in the regime $h/R \ll 1$ and $\rho V h/\mu \ll 1$.

   (a) **(5 marks)** Explain the significance of the two restrictions that specify the regime in which the motion is to be analysed.

   (b) **(10 marks)** Show that the radial pressure gradient in the air under the disk satisfies

   $$\frac{\partial p}{\partial r} = -\frac{6\mu r V}{h^3}.$$

   (c) **(10 marks)** Calculate the pressure, the force exerted on the disk and deduce that the distance h at which the disk is in equilibrium is given by

   $$h = R\left(\frac{3\pi\mu R V}{2Mg}\right)^{1/3}.$$

4. Incompressible fluid of density ρ and kinematic viscosity ν flows steadily past a streamlined two-dimensional body of characteristic dimension D. Far upstream of the body, the velocity field is uniform $\mathbf{u} = U\hat{\mathbf{x}}$. Downstream of the body there is a thin wake in the velocity field in which the streamwise velocity is given by $U - u_1(x, y)$, where $|u_1/U| \ll 1$.

   (a) **(3 marks)** Identify the Reynolds number for this flow. For what regime of Reynolds number is the wake thin?

   (b) **(8 marks)** Show that the equation governing the streamwise momentum is given to leading order by
   $$U\frac{\partial u_1}{\partial x} = \nu\frac{\partial^2 u_1}{\partial y^2}.$$
   What boundary conditions does u_1 satisfy?

   (c) **(4 marks)** Show that
   $$\int_{-\infty}^{\infty} u_1 \, \mathrm{d}y = Q,$$
   where Q is a constant.

   (d) **(10 marks)** Show that the velocity field is given by
   $$u_1 = \frac{Q}{x^{1/2}} f(y/x^{1/2}),$$
   where f is to be determined. How does the thickness of the wake vary with downstream distance? *Hint:* $\int_{-\infty}^{\infty} \mathrm{e}^{-s^2} \, \mathrm{d}s = \sqrt{\pi}$.

5. Incompressible fluid of dynamic viscosity μ resides in a porous medium bounded by two horizontal surfaces at $z = 0$ and $z = h$. The lower boundary is maintained at temperature $T = T_0 + \Delta T$, ($\Delta T > 0$), while the upper boundary is at temperature $T = T_0$. The local density of the fluid is proportional to the local temperature so that

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right),$$

where α is the positive-valued coefficient of expansion. The equation governing the advection and diffusion of the temperature field is given by

$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \kappa \nabla^2 T,$$

where κ is the diffusivity. The motion is assumed to be only two-dimensional so that the velocity field $\mathbf{u} = (u, w)$ is given by

$$u = -\frac{K_h}{\mu}\frac{\partial p}{\partial x} \qquad \text{and} \qquad w = -\frac{K_v}{\mu}\left(\frac{\partial p}{\partial z} + \rho g\right),$$

where K_h and K_v are the constant horizontal and vertical permeabilities, p denotes the pressure and g denotes gravitational acceleration.

(a) **(3 marks)** Find the steady temperature, density and pressure when there is no fluid flow.

(b) **(10 marks)** Now introduce perturbations to the static state found in (a) and show that after linearising the governing equations that the vertical velocity satisfies

$$\left(K_h \frac{\partial^2}{\partial x^2} + K_v \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) w = \frac{K_v K_h \rho_0 g \alpha \Delta T}{\mu h}\frac{\partial^2 w}{\partial x^2}. \tag{1}$$

(c) **(7 marks)** Now examine the neutral mode for (1) of the form $w = \mathrm{e}^{ikx}\sin(n\pi z/h)$ to deduce the following condition for marginal linear stability

$$Ra = \frac{\left((kh)^2 + (n\pi)^2\right)\left((kh)^2 + \Gamma(n\pi)^2\right)}{(kh)^2},$$

where $\Gamma = K_v/K_h$ and Ra is a dimensionless parameter that is to be determined.

(d) **(5 marks)** Find the minimum value of Ra for the onset of linear instability.

End of examination.