UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ADVANCED FLUID DYNAMICS

MATH M0600

(Paper Code MATH-M0600)

January 2015, 3 hours

This paper contains **five** *questions*
A candidate's **FOUR** *best answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Candidates may bring into the examination their lecture notes and material distributed during the course (printed or handwritten), but may not bring in textbooks.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Incompressible viscous fluid flows between two rigid planes located at $y = -a$ and $y = a$, driven by a constant pressure gradient, $-G\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a unit vector parallel to the planes and $G > 0$. Fluid in the region $y < 0$ has dynamic viscosity μ_1 and fluid in $y > 0$ has dynamic viscosity μ_2. The interface between the fluids remains at $y = 0$.

   (a) **(4 marks)**
   Simplify the Navier-Stokes equations to derive a governing differential equation for the steady unidirectional flow $\mathbf{u} = u(y)\hat{\mathbf{x}}$. What boundary conditions must this velocity field satisfy?

   (b) **(7 marks)**
   Find the flow field and sketch the flow profile when (i) $\mu_1 = \mu_2$ and (ii) $\mu_1 \ll \mu_2$.

   (c) **(6 marks)**
   Calculate the shear stress, $\boldsymbol{\tau}_1$, exerted on the boundary $y = -a$ by the fluid and the shear stress, $\boldsymbol{\tau}_2$, exerted on the boundary $y = a$ by the fluid. Evaluate $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ and explain why this is independent of the viscosities.

   (d) **(8 marks)**
   Calculate the volume fluxes of fluid per unit width, q_1 and q_2, in the regions $-a < y < 0$ and $0 < y < a$, respectively. Show that $Q = q_1 + q_2$ is given by

$$Q = \frac{Ga^3}{12(\mu_1 + \mu_2)} \left(14 + \frac{\mu_2}{\mu_1} + \frac{\mu_1}{\mu_2} \right).$$

   Evaluate Q when (i) $\mu_1 = \mu_2$ and (ii) $\mu_1 \ll \mu_2$.

2. (a) **(4 marks)**
A rigid sphere of radius a translates with velocity $\mathbf{U}$ through an unbounded fluid that is at rest at infinity. The force per unit area on the sphere, $\sigma_{ij}n_j$, where σ_{ij} is the stress tensor and n_j the unit normal to the surface of the sphere, takes the uniform value $3\mu\mathbf{U}/(2a)$. Hence show that the drag exerted on the sphere is $6\pi\mu a\mathbf{U}$.

(b) **(10 marks)**
Show that two Stokes flows with velocity fields $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ and associated stress tensors $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ satisfy

$$\nabla_j \left(\sigma_{ij}^{(1)} u_i^{(2)} - \sigma_{ij}^{(2)} u_i^{(1)} \right) = 0. \tag{1}$$

Hence deduce that

$$\int_S \sigma_{ij}^{(1)} u_i^{(2)} n_j \, \mathrm{d}S = \int_S \sigma_{ij}^{(2)} u_i^{(1)} n_j \, \mathrm{d}S. \tag{2}$$

(c) **(11 marks)**
A rigid sphere of radius a, free of external forces, is placed with its centre at the origin of an unbounded Stokes flow, given in the absence of the sphere by $\mathbf{u}^*(\mathbf{x})$. The flow with the particle present is denoted by $\mathbf{u}$ and the perturbation that it introduces is $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$.

Employ (2) using the flow field given in (a) and the perturbation field, $\mathbf{v}$, on the assumption that it tends to zero sufficiently rapidly at infinity, to show that the instantaneous velocity at the centre of the sphere is given by

$$\frac{1}{4\pi a^2} \int_{r=a} \mathbf{u}^*(\mathbf{x}) \, \mathrm{d}S.$$

3. Incompressible fluid of density ρ and dynamic viscosity μ is forced to flow between two stationary circular cylinders of radius a, the centres of which are located at $y = \pm(a+b/2)$, by an imposed difference in pressure. The x-axis is perpendicular to the line joining the centres and the origin is located at the midpoint between the cylinders. The motion is analysed in the regime $b/a \ll 1$ and $\Delta p \ll \mu^2 a/(\rho b^3)$.

   (a) **(4 marks)**
   Show that to leading order in x/a the gap between the surfaces of the cylinders is given by
   $$h(x) = b + \frac{x^2}{a}.$$

   (b) **(8 marks)**
   Show that the volume flux of fluid per unit width, q, passing between the cylinders is given by
   $$q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x},$$
   where $\partial p/\partial x$ denotes the pressure gradient.

   (c) **(8 marks)**
   The pressure difference between the fluid far upstream ($x \to -\infty$) and far downstream ($x \to \infty$) is given by Δp. Thus show that the volume flux of fluid per unit width is given by
   $$q = \frac{2\Delta p b^{5/2}}{9\pi\mu a^{1/2}}.$$

   (d) **(5 marks)**
   Now suppose that the cylinders rotate about their axes, such that the angular speed of the cylinder with axis at $y = (a+b/2)$ is Ω, while the angular speed of the cylinder with axis at $y = -(a+b/2)$ is $-\Omega$. Show that the additional volume flux per unit width is given by
   $$\frac{4}{3}ba\Omega.$$

Continued...

4. Incompressible fluid of density ρ and kinematic viscosity ν is extracted from the tip of a cone. In terms of spherical polar coordinates (r, θ, ϕ), the tip of the cone is located at the origin, its symmetry axis is along $\theta = 0$ and its surface is at $\theta = \alpha$. The total volume flux of fluid through the tip is constant and denoted by Q. Sufficiently close to the tip, but outside of relatively thin boundary layers at $\theta = \alpha$, the velocity field is approximately given by $\mathbf{u} = -(A/r^2)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit radial vector and A is a constant.

   (a) **(6 marks)**
   Show that
   $$A = \frac{Q}{2\pi(1 - \cos\alpha)}.$$

   (b) **(3 marks)**
   Construct a Reynolds number for the flow and show that the Reynolds number becomes large as the origin is approached.

   (c) **(3 marks)**
   Close to the edge of the cone $\theta = \alpha$, a boundary layer develops. Write down the boundary layer equations for in this region, using a local Cartesian coordinates x and y that are parallel and perpendicular to the boundary, respectively.

   (d) **(6 marks)**
   By finding the appropriate balance of terms show that the width of the boundary layer $\delta(x) = (\nu x^3/A)^{1/2}$ and hence that a streamfunction for the motion is given by
   $$\psi = \left(\frac{A\nu}{x}\right)^{1/2} F\left(\frac{y}{\delta(x)}\right), \tag{3}$$
   where F is an undetermined function.

   (e) **(7 marks)**
   Derive the governing equation satisfied by F and give appropriate boundary conditions.

5. Inviscid incompressible fluid of uniform density flows steadily with velocity $\mathbf{u}_0 = U_0(z)\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ denotes a vector parallel with the x-axis.

   (a) **(6 marks)**
   The steady flow is perturbed so that $\mathbf{u} = \mathbf{u}_0 + (\hat{u}(z), 0, \hat{w}(z))\,\mathrm{e}^{\mathrm{i}k(x-ct)}$. Derive the linearised equations governing the perturbation to the velocity field and show that

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} - k^2\hat{w} - \frac{1}{U_0 - c}\frac{\mathrm{d}^2U_0}{\mathrm{d}z^2}\hat{w} = 0, \tag{4}$$

   where a prime denotes differentiation with respect to z.

   (b) **(6 marks)**
   When $U_0(z) = \begin{cases} 1, & z > 1, \\ z, & |z| \leq 1, \\ -1, & z < -1, \end{cases}$ show that the boundary conditions on the linearised fields demand that both

$$\hat{w} \qquad \text{and} \qquad \hat{w}\frac{\mathrm{d}U_0}{\mathrm{d}z} - (U_0 - c)\frac{\mathrm{d}\hat{w}}{\mathrm{d}z}$$

   are continuous at $z = \pm 1$.

   (c) **(13 marks)**
   Show that if $k > 0$ and the solution for $\hat{w}$ is given by $\hat{w}(z) = \begin{cases} A\mathrm{e}^{-kz}, & z > 1, \\ \mathrm{e}^{-kz} + B\mathrm{e}^{kz}, & |z| \leq 1, \\ D\mathrm{e}^{kz}, & z < -1, \end{cases}$

   where A, B and D are constants to be determined, then

$$c^2 = \frac{(1-2k)^2 - \mathrm{e}^{-4k}}{4k^2}.$$

   For what values of the wavenumber, k, is the flow $U_0(z)$ linearly unstable?

End of examination.