UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci.

ADVANCED FLUID DYNAMICS

MATH M0600

(Paper Code MATH-M0600)

January 2016, 3 hours

This paper contains **four** *questions*
All **FOUR** *answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Candidates may bring into the examination their lecture notes and material distributed during the course (printed or handwritten), but may not bring in textbooks.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Incompressible viscous fluid flows between two vertical rigid planes located at $x = -a$ and $x = b$, driven by a constant pressure gradient, $-G\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector parallel to the planes and $G > 0$. The fluid is of dynamic viscosity μ and density ρ_1 and experiences the gravitational body force $-\rho_1 g\hat{\mathbf{z}}$.

   (a) **(6 marks)**
   Simplify the Navier-Stokes equations to derive a governing differential equation for the steady unidirectional flow $\mathbf{u} = w(x)\hat{\mathbf{z}}$. Show that the volume flux of fluid per unit width is given by $(G - \rho_1 g)(a + b)^3/(12\mu)$.

   Now suppose that fluid of density ρ_1 occupies $-a < x < 0$, while fluid of density ρ_2 occupies $0 < x < b$; the dynamic viscosity of both fluids is μ. The motion is still driven by a constant pressure gradient, $-G\hat{\mathbf{z}}$, where it is convenient to write $G = (\rho_1 + \rho_2)g/2 + \lambda\mu$, and both fluids flow steadily parallel to the boundaries.

   (b) **(10 marks)**
   Denoting $m = (\rho_2 - \rho_1)g/(2\mu)$, show that the flow field in $-a < x < 0$ is given by

$$w_1 = -\frac{(m+\lambda)}{2}x^2 + Ax + \frac{ab}{2(a+b)}\left(\lambda(a+b) + m(a-b)\right),$$

   and in $0 < x < b$ is given by

$$w_2 = \frac{(m-\lambda)}{2}x^2 + Ax + \frac{ab}{2(a+b)}\left(\lambda(a+b) + m(a-b)\right),$$

   where A is to be determined.

   (c) **(9 marks)**
   Find the net flux per unit width of fluid travelling along the channel and hence show that the net flux vanishes when the pressure gradient, $G = G^*$, is given by

$$G^* = \frac{(\rho_1+\rho_2)g}{2} + \frac{m\mu(b-a)(a^2+4ab+b^2)}{(a+b)^3}.$$

2. Incompressible fluid of density ρ and dynamic viscosity μ flows in the narrow gap between the plane $z = 0$ and an inclined rigid plate, settling towards the plane with speed V. At the instant to be analysed in this question, the settling plate lies along $z = h(x) = h_0 + \Delta h x/L$ for $-L \le x \le L$. There is no externally imposed pressure gradient on the motion.

   (a) **(4 marks)**
   Identify the conditions that must be satisfied if lubrication theory is to be used to model the flow.

   (b) **(6 marks)**
   In the lubrication regime, show that the volume flux of fluid per unit width flowing between the plate and the underlying plane is given by

   $$q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}.$$

   (c) **(5 marks)**
   Show that mass conservation demands $\mathrm{d}q/\mathrm{d}x = V$ and hence deduce that

   $$q(L) - q(-L) = 2VL.$$

   (d) **(10 marks)**
   By using the condition that there is no difference in pressure between $x = -L$ and $x = L$, deduce another condition linking $q(L)$ and $q(-L)$ and thus show that

   $$q(\pm L) = \pm VL\left(1 \pm \frac{\Delta h}{h_0}\right).$$

3. Incompressible fluid of density ρ and kinematic viscosity ν flows steadily at high Reynolds number above the semi-infinite plane $y = 0$ $(x > 0)$. Far from the plane, the flow field is parallel with the boundary and given $u = U(x)$.

   (a) **(8 marks)**
   Explain why a boundary layer arises in this flow. Write down and justify the boundary layer equations governing the velocity $\mathbf{u} = (u, v)$ and pressure, p, fields.

   The flow far from the boundary is given by $U = -\Gamma/x$, where Γ is a constant.

   (b) **(3 marks)**
   By estimating the magnitude of the terms in the boundary layer equation, show that the size of the boundary layer, $\delta(x)$, is given by

$$\delta = \left(\frac{\nu}{\Gamma}\right)^{1/2} x.$$

   (c) **(8 marks)**
   Introduce the streamfunction $\psi = -(\nu\Gamma)^{1/2} f(\xi)$, where $\xi = y/\delta(x)$ and from the boundary layer equations, derive the following equation

$$f''' - (f')^2 + 1 = 0,$$

   where a prime denotes differentiation with respect to ξ. What boundary conditions must f satisfy?

   (d) **(6 marks)**
   Verify by substitution or otherwise that the solution for u may be written

$$u = -\frac{\Gamma}{x}\left(3\tanh^2(A + B\xi) - 2\right),$$

   where A and B are to be determined.

4. (a) **(7 marks)**
For an inviscid fluid of constant density, flowing with velocity field $\mathbf{u}$, derive the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}.\nabla\boldsymbol{\omega} = \boldsymbol{\omega}.\nabla\mathbf{u}.$$

How does this equation simplify for two-dimensional flows?

The rest of this question examines two-dimensional flows in terms of polar coordinates, with unit radial and angular vectors denoted by $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, respectively. A unit vector perpendicular to the (r, θ) plane is denoted $\hat{\mathbf{z}}$.

(b) **(2 marks)**
Show that the vorticity of the flow $\mathbf{u}_0 = V(r)\hat{\boldsymbol{\theta}}$ is given by

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rV).$$

(c) **(8 marks)**
Now suppose the flow $\mathbf{u}_0$ is perturbed such that $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1(r, \theta, t)$, where $|\mathbf{u_1}| \ll |V|$. By introducing a streamfunction such that

$$\mathbf{u}_1 = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{r}} - \frac{\partial\psi}{\partial r}\hat{\boldsymbol{\theta}},$$

show that the linearised vorticity equation governing the perturbation is given by

$$\left(\frac{\partial}{\partial t} + \frac{V}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) - \frac{1}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r}\frac{\partial\psi}{\partial\theta} = 0. \qquad (1)$$

(d) **(8 marks)**
Now substitute $\psi = \phi(r)\exp(st + \mathrm{i}n\theta)$ into (1). Hence on the assumption that $\phi(R_1) = \phi(R_2) = 0$, deduce

$$s_r \int_{R_1}^{R_2} \frac{\mathrm{d}\Omega}{\mathrm{d}r}\left|\frac{\phi}{s + \mathrm{i}nV/r}\right|^2 \mathrm{d}r = 0,$$

where $s = s_r + \mathrm{i}s_i$. Thus deduce a necessary condition for the linear instability of the flow $\mathbf{u}_0$ to two-dimensional perturbations.

Hint: You are given curl $\left(a\hat{\mathbf{r}} + b\hat{\boldsymbol{\theta}}\right) = \left(\frac{1}{r}\frac{\partial}{\partial r}(rb) - \frac{1}{r}\frac{\partial a}{\partial\theta}\right)\hat{\mathbf{z}}.$

End of examination.