UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci.

ADVANCED FLUID DYNAMICS

MATH M0600

(Paper Code MATH-M0600)

January 2017, 3 hours

This paper contains **four** *questions*
All **FOUR** *answers will be used for assessment.*

Calculators are **not** *permitted in this examination.*

Candidates may bring into the examination their lecture notes and material distributed during the course (printed or handwritten), but may not bring in textbooks.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Viscous fluid of dynamic viscosity, μ, resides between two long, concentric cylinders of radii a and b ($b < a$), which are held so that their axes are horizontal. The fluid is set in motion by the viscous stresses exerted on it by moving the outer cylinder with velocity, V_2 and the inner cylinder with velocity V_1. Both velocities are directed along the axis of the cylinders and in parts (a)-(c) there is no imposed pressure gradient.

   (a) **(5 marks)**
   Find the steady velocity field $\mathbf{u} = w(r)\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ denotes a unit vector along the axis of the cylinders and r is the radial distance from the axis.

   (b) **(6 marks)**
   Show that the rate at which energy is dissipated by the viscous processes per unit length along the axis of the cylinders is given by

$$2\pi\mu\frac{(V_2 - V_1)^2}{\log(a/b)}.$$

   (c) **(7 marks)**
   If the net volume flux of fluid transported along the annulus vanishes, show that the outer cylinder must be moved at velocity

$$\frac{V_2}{V_1} = \frac{R^2 - 1 - \log(R^2)}{R^2 - 1 - R^2\log(R^2)},$$

   where $R = a/b$. Hence, or otherwise, evaluate V_2 in the regime $|a/b - 1| \ll 1$.

   (d) **(7 marks)**
   Now suppose that there is a constant pressure gradient, $-G\hat{\mathbf{z}}$, within the fluid and that the outer cylinder is stationary ($V_2 = 0$). For what velocity V_1 is the rate of dissipation of energy by viscous processes minimised?

[Hint: for a flow with velocity field $\mathbf{u} = w(r)\hat{\mathbf{z}}$*, the axial component of the Navier-Stokes equation is given by*

$$\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) - \frac{\partial p}{\partial z} = 0.$$

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2. A steady two-dimensional flow of viscous fluid of constant density, ρ, and viscosity, μ, is modelled in terms of a streamfunction, $\psi(r, \theta)$, which is a function of plane polar coordinates (r, θ).

   (a) **(2 marks)**
   State the relationship between the velocity field and the streamfunction and show that the vorticity is given by $\omega\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector perpendicular to the plane and

   $$\omega = -\nabla^2\psi.$$

   (b) **(3 marks)**
   When inertial processes are negligible, show that the streamfunction satisfies the bi-harmonic equation

   $$\nabla^4\psi = 0.$$

Now consider a viscously dominated flow past a stationary circular disk of radius a. Far from the disk the flow field tends to a steady straining flow given in Cartesian coordinates relative to the origin located at the centre of the disk, by $\mathbf{u}_\infty = (Ex, -Ey)$, where E is a positive constant.

   (c) **(5 marks)**
   Find the streamfunction, $\psi_\infty(r, \theta)$, corresponding to the far-field flow in terms of plane polar coordinates.

   (d) **(10 marks)**
   Using your solution to (2c), make an appropriate assumption about the θ-dependence of the streamfunction for the viscously-dominated flow past a cylinder and hence calculate the streamfunction.

   (e) **(5 marks)**
   Show that the tangential stress exerted by the fluid on the surface of the disk is given by $4\mu E \sin 2\theta$.

[Hint: In plane polar coordinates,

$$\nabla . \mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \qquad \omega = \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$
$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Components of the rate of strain tensor are given by

$$e_{rr} = \frac{\partial u_r}{\partial r} \qquad e_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right).$$

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3. Fluid of kinematic viscosity ν and density ρ flows as an axisymmetric jet at high Reynolds number. Its velocity field is steady and modelled in terms of cylindrical polar coordinates (r, θ, z) such that $\mathbf{u} = u\hat{\mathbf{r}} + w\hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ are unit vector in the radial direction and along the symmetry axis, respectively. The flow is to be analysed using a boundary layer analysis in which the radial extent of the jet is much smaller than the axial lengthscale over which it evolves. There is no imposed pressure gradient.

   (a) **(4 marks)**
   Briefly justify why the axial component of the Navier-Stokes equations is given by

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$

   (b) **(4 marks)**
   Show that the momentum flux in the jet, given by

$$F(z) = 2\pi \int_0^\infty \rho w^2 r \, \mathrm{d}r$$

   is constant.

   (c) **(4 marks)**
   Use dimensional reasoning to relate the radial extent of the jet, $\delta(z)$, and the axial velocity scale, $W(z)$, to the momentum flux, F, the distance from the source, z and the density & viscosity of the fluid.

   (d) **(10 marks)**
   Seek a self-similar solution for the Stokes streamfunction for the flow of the form

$$\Psi(r, z) = \nu z f(\eta),$$

   where $\eta = r/\delta(z)$ and show that the dimensionless function $f(\eta)$ satisfies

$$ff' - \eta(f'^2 + ff'') = f' - \eta f'' + \eta^2 f'''.$$

   (e) **(3 marks)**
   What boundary conditions must be applied to $f(\eta)$ and what do they represent physically?

[Hint: In cylindrical polar coordinates for the axisymetric flow $\mathbf{u} = u\hat{\mathbf{r}} + w\hat{\mathbf{z}}$*, incompressibility is given by*

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0,$$

and the velocity field is related to the Stokes streamfunction by

$$u = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \qquad w = \frac{1}{r}\frac{\partial \Psi}{\partial r}.$$

Also the axial component of the Navier Stokes equation is given by

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right).$$

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4. (a) **(10 marks)**
Fluid flows as an inviscid vortex sheet, such that the velocity field is given by

$$\mathbf{u} = \begin{cases} U_0, & z > 0, \\ -U_0, & z < 0. \end{cases}$$

The interface at $z = 0$ is slightly perturbed so that it is given by $z = \eta(x, y, t) = \eta_0 \exp(\mathrm{i}(kx + ly) + st)$. Show that this flow is linearly unstable to three-dimensional disturbances to the motion which are proportional to $\exp(\mathrm{i}(kx + ly) + st)$ and that the growth rate is given

$$s = U_0 k.$$

Now consider a steady, viscous vortex sheet in the presence of a perpendicular straining flow so that the velocity field is of the form $\mathbf{u} = (U(z), Ey, -Ez)$ where $E > 0$ is a positive constant and $U \to \pm U_0$ as $z \to \pm\infty$.

(b) **(5 marks)**
Show that the vorticity of the flow has the form $\boldsymbol{\omega} = \omega\hat{\mathbf{y}} = (\partial U/\partial z)\hat{\mathbf{y}}$ and show that the equation governing the evolution of ω is given by

$$-Ez\frac{\partial \omega}{\partial z} = E\omega + \nu\frac{\partial^2 \omega}{\partial z^2},$$

where ν is the kinematic viscosity of the fluid.

(c) **(8 marks)**
Determine the steady vorticity, $\omega(z)$, and velocity, $U(z)$, fields. How is the thickness of the viscous vortex sheet, δ, related to the kinematic viscosity, ν and the strain rate, E?
[Hint: $\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \exp(-s^2)\,\mathrm{d}s$ *and* $\mathrm{erf}(z) \to 1$ *as* $z \to \infty$.*]*

(d) **(2 marks)**
Explain whether you anticipate long wavelength disturbances to this steady flow to be linearly unstable.

End of examination.