Fluid Dynamics 3 - 2015/2016

Jens Eggers

Preliminaries

Course information

- Lecturer: Prof. Jens Eggers, Room SM2.3
- Timetable: Weeks 1-12, Tuesday 1.00 (room 1.18 Queen's building), Thursday 1.00 (room 1.18 Queen's bulding) and Friday 10.00 (Physics 3.21). Course made up of 32 lectures.
- Drop-in sessions (formerly office hours): Monday 12:00 in my room, 2.3
- Prerequisites: Mechanics 1, APDE2 and Calc2. Need ideas from vector calculus, complex function theory, separation solutions and PDE's.
- Homework: Questions from 10 worksheets will be set and marked during the course. Homework sheets will be handed out each Friday, starting in the first week. Solutions to be returned the following Friday in the box marked "Fluids 3".
- Web: Standard unit description includes detailed course information. Lecture notes will be posted on Blackboard, along with homework and solution sheets.
- Lectures: There is no need to take notes during the lecture, as all material relevant for the exam will be put on the web. The main purpose of the lecture is the live development of the material, and a chance for you to ask questions!

Recommended texts

1. A.R. Paterson, *A First Course in Fluid Dynamics*, Cambridge University Press. (The recommended text to complement this course - costs $\approx$ £50 from Amazon; there are 6 copies in Queen's building Library and 3 copies in the Physics Library)
2. D.J. Acheson, *Elementary Fluid Dynamics.* Oxford University Press
3. L.D. Landau and E.M. Lifshitz, *Fluid Mechanics.* Butterworth Heinemann

Films

There is a very good series of educational films on Fluid Mechanics available on YouTube, produced by the National Committee for Fluid Mechanics Films in the US in the 1960's. Each film is also accompanied by a set of notes. I recommend them highly, and will point out the appropriate ones throughout this course.

The following 3 sections are useful for the course. For the purposes of an examination, I would expect you to know the definition of div, grad, curl and the Laplacian in Cartesians and grad and the Laplacian in plane polars. Other definitions would be provided.

Revision of vector operations

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be Cartesian vectors. Let $\phi(\mathbf{r})$ be a scalar function and $\mathbf{f}(\mathbf{r}) = (f_1(\mathbf{r}), f_2(\mathbf{r}), f_3(\mathbf{r}))$ a vector field of position $\mathbf{r} = (x, y, z) \equiv (x_1, x_2, x_3)$. Then

- The dot product is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$
- The cross (or vector) product is $\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{r}} + (u_3 v_1 - u_1 v_3)\hat{\mathbf{y}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{z}}$
- The gradient is $\boldsymbol{\nabla}\phi = \left(\dfrac{\partial \phi}{\partial x_1}, \dfrac{\partial \phi}{\partial x_2}, \dfrac{\partial \phi}{\partial x_3}\right)$
- The divergence is $\boldsymbol{\nabla} \cdot \mathbf{f} = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} + \dfrac{\partial f_3}{\partial x_3}$
- The curl is $\boldsymbol{\nabla} \times \mathbf{f} = \left(\dfrac{\partial f_3}{\partial x_2} - \dfrac{\partial f_2}{\partial x_3}\right)\hat{\mathbf{r}} + \left(\dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_3}{\partial x_1}\right)\hat{\mathbf{y}} + \left(\dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2}\right)\hat{\mathbf{z}}$
- The Laplacian is $\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \dfrac{\partial^2 \phi}{\partial x_1^2} + \dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2}$
- $(\mathbf{f}\cdot\boldsymbol{\nabla})\mathbf{f} = \left(f_1\frac{\partial f_1}{\partial x_1} + f_2\frac{\partial f_1}{\partial x_2} + f_3\frac{\partial f_1}{\partial x_3}\right)\hat{\mathbf{r}} + \left(f_1\frac{\partial f_2}{\partial x_1} + f_2\frac{\partial f_2}{\partial x_2} + f_3\frac{\partial f_2}{\partial x_3}\right)\hat{\mathbf{y}} + \left(f_1\frac{\partial f_3}{\partial x_1} + f_2\frac{\partial f_3}{\partial x_2} + f_3\frac{\partial f_3}{\partial x_3}\right)\hat{\mathbf{z}}$

Formulae in cylindrical polar coordinates

Coordinate system is $\mathbf{r} = (r, \theta, z)$ where the relationship to Cartesians is $x = r\cos\theta$, $y = r\sin\theta$. The unit vectors are $\hat{\mathbf{r}} = \hat{\mathbf{r}}\cos\theta + \hat{\mathbf{y}}\sin\theta$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{y}}\cos\theta$ and $\hat{\mathbf{z}}$. In the following, $\mathbf{f} = (f_r, f_\theta, f_z) \equiv f_r\hat{\mathbf{r}} + f_\theta\hat{\boldsymbol{\theta}} + f_z\hat{\mathbf{z}}$.

- The gradient is $\boldsymbol{\nabla}\phi = \hat{\mathbf{r}}\dfrac{\partial \phi}{\partial r} + \hat{\boldsymbol{\theta}}\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} + \hat{\mathbf{z}}\dfrac{\partial \phi}{\partial z}$
- The divergence is $\boldsymbol{\nabla} \cdot \mathbf{f} = \dfrac{1}{r}\dfrac{\partial (r f_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{\partial f_z}{\partial z}$
- The curl is $\boldsymbol{\nabla} \times \mathbf{f} = \dfrac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ f_r & r f_\theta & f_z \end{vmatrix}$.
- The Laplacian is $\boldsymbol{\nabla}^2\phi = \dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} + \dfrac{\partial^2 \phi}{\partial z^2}$
- $(\mathbf{f} \cdot \boldsymbol{\nabla})\mathbf{f} = \left(f_r\frac{\partial f_r}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_r}{\partial \theta} + f_z\frac{\partial f_r}{\partial z} - \frac{f_\theta^2}{r}\right)\hat{\mathbf{r}} + \left(f_r\frac{\partial f_\theta}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_\theta}{\partial \theta} + f_z\frac{\partial f_\theta}{\partial z} + \frac{f_r f_\theta}{r}\right)\hat{\boldsymbol{\theta}} + \left(f_r\frac{\partial f_z}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_z}{\partial \theta} + f_z\frac{\partial f_z}{\partial z}\right)\hat{\mathbf{z}}$

Formulae in spherical polar coordinates

Coordinate system is $\mathbf{r} = (r, \theta, \varphi)$ where the relationship to Cartesians is $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.

The unit vectors are $\hat{\mathbf{r}} = \hat{\mathbf{r}}\sin\theta\cos\varphi + \hat{\mathbf{y}}\sin\theta\sin\varphi + \hat{\mathbf{z}}\cos\theta$, $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta\cos\varphi + \hat{\mathbf{y}}\cos\theta\sin\varphi - \hat{\mathbf{z}}\sin\theta$ and $\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{r}}\sin\varphi + \hat{\mathbf{y}}\cos\varphi$.

In the following, $\mathbf{f} = (f_r, f_\theta, f_\varphi) \equiv f_r\hat{\mathbf{r}} + f_\theta\hat{\boldsymbol{\theta}} + f_\varphi\hat{\boldsymbol{\varphi}}$.

- The gradient is $\boldsymbol{\nabla}\phi = \hat{\mathbf{r}}\dfrac{\partial\phi}{\partial r} + \hat{\boldsymbol{\theta}}\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} + \hat{\boldsymbol{\varphi}}\dfrac{1}{r\sin\theta}\dfrac{\partial\phi}{\partial\varphi}$
- The divergence is $\boldsymbol{\nabla}\cdot\mathbf{f} = \dfrac{1}{r^2}\dfrac{\partial(r^2 f_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta f_\theta)}{\partial\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial f_\varphi}{\partial\varphi}$
- The curl is $\boldsymbol{\nabla}\times\mathbf{f} = \dfrac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ f_r & rf_\theta & r\sin\theta f_\varphi \end{vmatrix}.$
- The Laplacian is $\boldsymbol{\nabla}^2\phi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\phi}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2\phi}{\partial\varphi^2}$
- $(\mathbf{f}\cdot\boldsymbol{\nabla})\mathbf{f} = \left(f_r\frac{\partial f_r}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_r}{\partial\theta} + \frac{f_\varphi}{r\sin\theta}\frac{\partial f_r}{\partial\varphi} - \frac{f_\theta^2+f_\varphi^2}{r}\right)\hat{\mathbf{r}} + \left(f_r\frac{\partial f_\theta}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{f_\varphi}{r\sin\theta}\frac{\partial f_\theta}{\partial\varphi} + \frac{f_r f_\theta}{r} - \frac{f_\varphi^2\cot\theta}{r}\right)\hat{\boldsymbol{\theta}} + \left(f_r\frac{\partial f_\varphi}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_\varphi}{\partial\theta} + \frac{f_\varphi}{r\sin\theta}\frac{\partial f_\varphi}{\partial\varphi} + \frac{f_r f_\varphi}{r} + \frac{f_\theta f_\varphi\cot\theta}{r}\right)\hat{\boldsymbol{\varphi}}$

1 Introduction & Basic ideas

1.1 What is a fluid ?

In this course we will treat the laws governing the motion of liquids and gases. A liquid or gas is characterized by the fact that there is no preferred rest state for the parts it is composed of. If a hole is made in a water bottle, the water will flow out. If a drop is placed on a solid surface, it will spread. By contrast, a solid retains a memory of its original state. If one deforms a piece of metal, it will relax back to its original state once the force is no longer applied. Collectively, we will call liquids or gases, which share this fluid property, "fluids".

Another important property of liquids and gases is that they are "featureless". Viewed from a particular point in space, all directions are equivalent. Solids, on the other hand, often have an internal (lattice) structure. As a result, it makes a difference in which direction they are deformed relative to their internal structure.

Fluid dynamics is an example of 'continuum' mechanics:

Definition 1.1.1 (Continuum) *A <u>continuum</u> is any medium whose state at a given instant can be described in terms of a set of continuous functions of position* $\mathbf{r} = (x, y, z)$. *E.g. Density,* $\rho(\mathbf{r}, t)$*, velocity,* $\mathbf{u}(\mathbf{r}, t)$*, temperature* $T(\mathbf{r}, t)$*. (These functions usually also depend upon time,* t*).*

We know that this description fails if we observe matter on small enough length scales: e.g. typical molecule size ($\sim 10^{-9}$m) or typical mean free path in a gas ($\sim 10^{-7}$m). The miracle is that on a scale only slightly larger than that, all microscopic features can be ignored, and we end up with a "universal" description of all things fluid. We will also ignore all effects of incompressibility, so for the purposes of this course, liquids and gases are the same thing. We will almost always speak of a fluid, but which can mean either a liquid or a gas for the purposes of this course.

The motion of liquids and gases is governed by the same underlying principles.

1.2 What we would like to do

In this lecture course, we will first develop an equation of motion for the velocity field $\mathbf{u}(\mathbf{r}, t)$, which gives the fluid velocity at any instant in time t, everywhere in space. This equation (or set of equations) will necessarily have the form of a *partial differential equation.* **It will be based on Newton's equations of motion, but for a continuum of particles, distributed over space. One effect we will neglect is the friction between fluid particles. The mathematical idealization of this situation is called an "Ideal Fluid".**

With the equations in hand, it is down to our ability to deal with the mathematical complexities of solving a partial differential equation (PDE) to solve physical problems. Some examples of problems dealt with rather successfully using the concept of ideal fluids are the following:

Figure 1: A jet of water from a bottle.

Figure 1 shows a jet of water from a bottle. Both the efflux of the water and the trajectory of the resulting jet are well described by ideal fluid theory, which we will describe.

Figure 2: An airplane is held up by the lift generated by the wings.

Another spectacular success is the theory of flight. The ideal flow of air around a wing is able to describe the lift necessary for flight, and much more. What is much more difficult is the theory of *drag*. Inviscid theory suggests that there should be no energy cost to flight at all!

Figure 3: Water waves on the surface of a lake.

The last topic we will be covering in this course is the huge area of water waves, see Fig 3. This includes waves from the scale of millimetres up to huge tsunami waves. The absence of any solid boundary results in very little friction, so the ideal theory works very well.

1.3 Lagrangian and Eulerian descriptions of the flow

We now begin to develop a dynamical description of fluid flow, which will lead us to formulate a PDE for fluid motion, known as the **Euler equation**. Before we can do that, we must understand the motion of fluids a little better. The description of motion is called *kinematics*. In this chapter, we will deal with kinematics. I encourage you to look at the film "Fluid Mechanics (Eulerian and Lagrangian description) parts 1-3" on YouTube.

There are two very different ways of describing fluid motion, known as the Eulerian and Lagrangian description. Ultimately, they are equivalent, as they describe the same thing. However, they serve different purposes, so we need them both.

Definition 1.3.1 (Eulerian description of the flow) . *This is what the stationary observer sees. Choose a fixed point,* $\mathbf{r}$ *to measure, for e.g. the velocity* $\mathbf{u}(\mathbf{r}, t)$. *This provides a spatial distribution of the flow at each instant in time. This is the way continuum equations are usually formulated, and our equation of motion will indeed be an equation for the Eulerian field* $\mathbf{u}(\mathbf{r}, t)$. *If the flow is* <u>*steady*</u>*, then* $\mathbf{u}$ *does not depend on time, t:* $\mathbf{u} = \mathbf{u}(\mathbf{r})$.

Why do we need anything else? The reason is that to make contact with Newton's equations, we need to describe the flow as a moving particle would see it. This is the

Definition 1.3.2 (Lagrangian description of the flow) *The observer moves with the fluid. Choose a fluid particle (for example, we can place a small drop of ink in the fluid),* *and follow it through the fluid. Measuring its velocity at a given time, t gives its 'Lagrangian velocity'. Now we describe the whole velocity field this way, by labeling all material points. A convenient way of doing so is to choose an initial time t_0, and to label all fluid particle by their position $\mathbf{r} = \mathbf{a}$ at that time. Then at time $t > t_0$, the particle is at*

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t), \ \textit{where}\ \mathbf{r}(\mathbf{a}, t_0) = \mathbf{a}.$$

Of course, $\mathbf{r}(\mathbf{a}, t)$ is very interesting in its own right. For example, it describes the course of a balloon, launched at time t_0 and at position $\mathbf{a}$ into the atmosphere. The Lagrangian velocity is defined as

$$\mathbf{v}(\mathbf{a}, t) = \left.\frac{d\mathbf{r}}{dt}\right|_{\mathbf{a} fixed}. \tag{1}$$

By definition, it is the velocity of a particle going with the flow. This is precisely how velocity is defined in Newtonian Mechanics. As we said earlier, the Eulerian and the Lagrangian velocity fields contain the same information; the relation between the two is:

$$\mathbf{v}(\mathbf{a}, t) = \mathbf{u}(\mathbf{r}(\mathbf{a}, t), t). \tag{2}$$

An example: Consider logs flowing along a narrowing section of river. A fixed observer measures the velocity by observing the velocity of logs at a given point in space. By observing many logs at different positions, he will be able to obtain the entire Eulerian velocity field $\mathbf{u}(\mathbf{r}, t)$. Unless the flow conditions are changing, this field will be time-independent, $\mathbf{u}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r})$. Now imagine each log being ridden by a moving observer, each of whom reports his velocity as time goes by. If all observers are labeled by their position $\mathbf{a}$ at some reference time t_0, this will produce the Lagrangian velocity field $\mathbf{v}(\mathbf{a}, t)$. The logs travel with the fluid and will see the flow accelerating, as the river bed becomes narrower. Thus $\mathbf{v}(\mathbf{a}, t)$ is manifestly time-dependent although the Eulerian field is not!

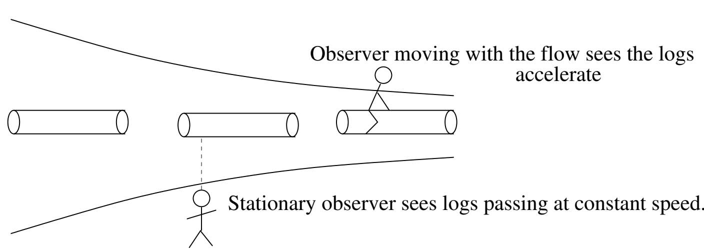


Figure 4: Stationary and co-moving observers of a flow.

Definition 1.3.3 (Two-dimensional flow) *A flow is two-dimensional if it is independent of one of its components (in some fixed frame of reference). E.g.* $\mathbf{u} = (u, v, 0)$.

Example 1.3.4 (Hyperbolic flow) *Let us consider a very simple model flow for the river shown above. Consider the two-dimensional, stationary Eulerian velocity field* $\mathbf{u} = (u, v, 0)$, *defined by*

$$u = kx, \quad v = -ky. \tag{3}$$

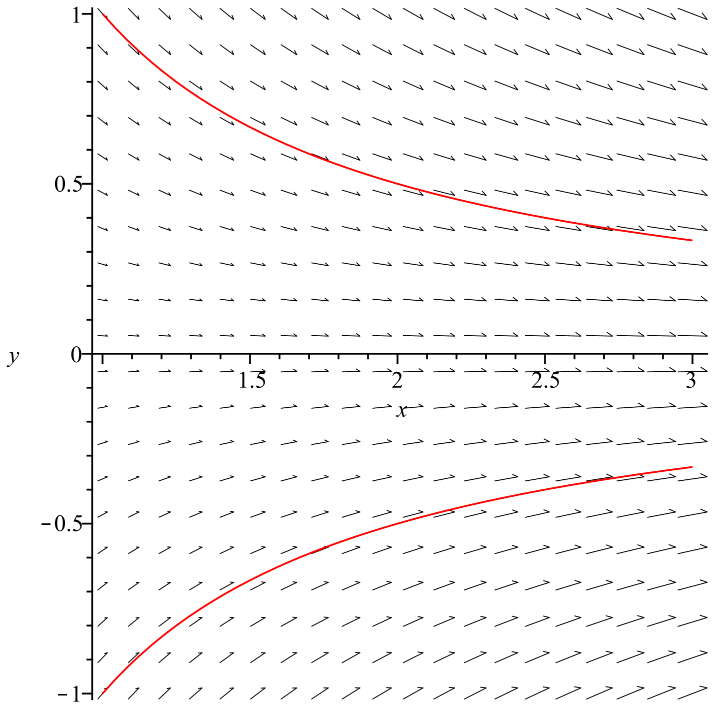


Figure 5: Local velocities of the flow (3). The red lines are streamlines, and denote the bank of the river.

As can be seen in Fig. 5, the flow speeds up as the river contracts, which we can think of as being confined by the river banks, shown as the red lines.

1.4 Particle paths and streamlines

Definition 1.4.1 (Pathlines) *The particle paths (pathlines) are the paths followed by individual particles. They are determined by the solution of the differential equation:*

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t), \qquad \text{with initial condition } \mathbf{r}(t_0) = \mathbf{a} = (a_1, a_2, a_3) \tag{4}$$

where $\mathbf{u}$ *is assumed given.*

The system of equations (4) specifies a unique curve. For some (simple) $\mathbf{u}$, can be integrated using elementary methods, but in general not. Let $\mathbf{u} = (u(\mathbf{r}, t), v(\mathbf{r}, t), w(\mathbf{r}, t))$, then in components, (4) is

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x, y, z, t) \\ \frac{dy}{dt} &= v(x, y, z, t) \\ \frac{dz}{dt} &= w(x, y, z, t) \end{aligned}\right\}$$

with $x(t_0) = a_1$, $y(t_0) = a_2$, $z(t_0) = a_3$.

Example 1.4.2 (Pathlines of a steady flow) *Find the pathlines for the flow field (3).*

Then the equations become

$$\dot{x} = kx, \quad \dot{y} = -ky.$$

The initial positions $\mathbf{a} = (a_1, a_2)$ at time $t_0 = 0$, are

$$x(t) = a_1 e^{kt}, \quad y(t) = a_2 e^{-kt},$$

which describes the path of particles in the flow.

To find the shape of the curve followed by a particle in Example 1.4.2, we eliminate time between $x(t)$ and $y(t)$, to find

$$xy = a_1 a_2, \tag{5}$$

which is the equation of a hyperbola. An example is shown in Fig. 5.

Using the relation (2) between Eulerian and Lagrangian fields, we find

$$\mathbf{v}(\mathbf{a}, t) = \left(ka_1 e^{kt}, -ka_2 e^{-kt}\right).$$

In particular, the Lagrangian velocity field is indeed time-dependent, while the Eulerian field was steady.

Now let us consider a case where the Eulerian field is also time dependent.

Example 1.4.3 (Pathlines of an unsteady flow)

Calculate the Pathlines of the two-dimensional flow

$$\mathbf{u} = (1, t). \tag{6}$$

The equations of motion are

$$\dot{x} = 1, \quad \dot{y} = t,$$

with solution

$$x = x_0 + t, \quad y = y_0 + \frac{t^2}{2}.$$

For example the path of a particle with initial position $\mathbf{a} = (0, 1)$ would be $(x, y) = (t, 1 + t^2/2)$.

What is the shape of a particle path? Eliminating time, we find

$$y = y_0 + \frac{(x - x_0)^2}{2}, \tag{7}$$

which is a parabola, but with different coefficients, depending on the initial particle position.

Now we come to the definition of a streamline, which describes the geometry of a flow at a given time t_0. As a result, the pattern of streamlines will in general *change in time*.

Definition 1.4.4 (Streamline)

A <u>streamline</u> of a flow $\mathbf{u}(\mathbf{r}, t)$ *at a given instant in time* t_0*, is a curve which is everywhere parallel to* $\mathbf{u}(\mathbf{r}, t_0)$.

Thus, along a streamline,

$$\frac{1}{\lambda(s)}\frac{d\mathbf{r}}{ds} = \mathbf{u}(\mathbf{r}, t_0), \tag{8}$$

where s parameterizes the streamline $\mathbf{r}(s)$, and $\lambda(s)$ is an arbitrary function. This illustrates that the *magnitude* of $\mathbf{u}$ does not matter for the definition of the streamline, only its direction. To find all possible streamlines, we have to solve (8) with different initial conditions $\mathbf{r}(s=0) = \mathbf{r}_0$. Eliminating $\lambda(s)ds$ from (8), we find

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}(= \lambda(s)ds), \qquad t = t_0. \tag{9}$$

The function $\lambda(s)$ does not change the shape of the streamline, but only the parameterization. To complete the picture, we also add arrows pointing in the direction of the flow along a streamline (see Fig. 6 below).

The streamline pattern will in general be different for different times t_0, since the velocity field $\mathbf{u}(\mathbf{r}, t_0)$ is different. For a steady flow, the pattern is of course time-independent as well. Streamlines can be visualized by taking a short-time exposure of illuminated particles in a flow. Taking an arbitrary starting point, one always continues in the local flow direction.

Remark 1.4.5 (Streamlines and pathlines of steady flows)

For **steady flows***, streamlines and pathlines coincide. Since* $\mathbf{u}$ *does not depend on time, we can equate the left hand sides of the defining equations (8) and (4) to obtain*

$$\frac{d\mathbf{r}}{dt} = \frac{1}{\lambda(s)}\frac{d\mathbf{r}}{ds}.$$

Thus if $\lambda(s)$ *is chosen such that*

$$\frac{ds}{dt} = \frac{1}{\lambda(s)},$$

and thus

$$t = \int_{s_0}^{s} \lambda(s)ds,$$

the two definitions become identical. In other words, time is a particular parameterization of the streamline.

Example 1.4.6 (Streamlines of a steady flow)

As in Example 1.4.2, we take the flow $\mathbf{u} = (kx, -ky)$*.*

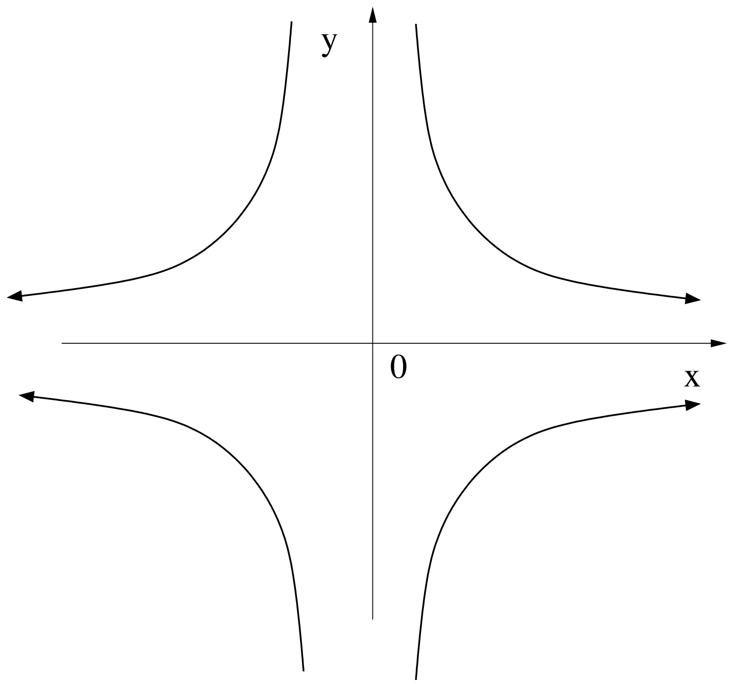


Figure 6: The streamlines of the flow (3). Arrows are drawn for the case $k > 0$.

Since this is a steady flow, we can eliminate t from the particle path to obtain (5), which must also be a streamline. Alternatively, we have

$$\frac{dx}{kx} = -\frac{dy}{kv}$$

according to (9). Integrating, we find $\ln|x| + \ln|y| = C$, or

$$|x||y| = e^C,$$

which is (5), the equation for a hyperbola. Any such hyperbola is a possible equation for the river bank, which must be a streamline of the flow, see Fig. 5.

Example 1.4.7 (Streamlines of an unsteady flow)

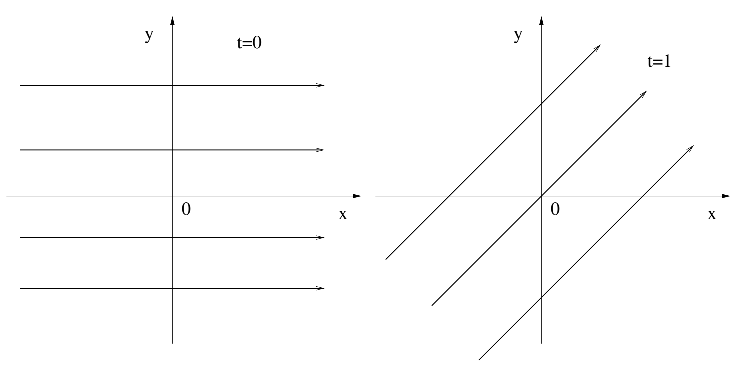


Figure 7: The streamlines of the flow (10). The pattern changes in time.

As in Example 1.4.3, we take $\mathbf{u} = (1, t)$.

Thus the equation (9) for the streamline becomes

$$dx = \frac{dy}{t},$$

with solution

$$y = tx + y_0. \tag{10}$$

Thus streamlines of the flow are straight lines, but whose slope depends on time, see Fig. 7. The streamlines are completely different from the pathlines (7) of the same flow, which are parabolae.

1.5 The Lagrangian derivative

(a.k.a. the <u>convective</u> derivative, or the <u>material</u> derivative).

We know how to measure the time derivative of a physical quantity associated with the fluid, for example that of the temperature $T(\mathbf{r}, t)$, at a fixed point in space (the Eulerian derivative); it is $\frac{\partial T}{\partial t}$. This quantity will describe the change of temperature at a fixed location, for example air temperature in Bristol. However, this quantity will not be a measure of how a mass of air heats up or becomes colder. The reason is that air is swept away by the prevailing flow field $\mathbf{u}(\mathbf{r}, t)$. In other words, to describe the change of temperature of a piece of air, we need to consider the <u>rate of change of</u> $T(\mathbf{r}, t)$, <u>following a fluid particle</u> with trajectory $\mathbf{r}(\mathbf{a}, t)$.

Definition 1.5.1 (Lagrangian time derivative)

The Lagrangian derivative, denoted as $\frac{DT}{Dt}$, *is the time derivative of a field* $T(\mathbf{r}, t)$ *along the trajectory of a fluid particle.*

Thus using the chain rule, we have

$$\frac{DT}{Dt} \equiv \frac{dT}{dt} = \frac{\partial T}{\partial t} + (\dot{\mathbf{r}} \cdot \boldsymbol{\nabla})T.$$

But according to (1) and (2), $\dot{\mathbf{r}} = \mathbf{v}(\mathbf{a}, t) = \mathbf{u}(\mathbf{r}, t)$, and so

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})T. \tag{11}$$

A particularly important example is the change in velocity (the acceleration) of a fluid particle, which we need to apply Newton's equations to fluid motion. Of course, the velocity is a vector quantity, which means we have to apply (11) to each component:

$$\frac{D\mathbf{u}}{Dt} = \left(\frac{Du_1}{Dt}, \frac{Du_2}{Dt}, \frac{Du_3}{Dt} \right).$$

The the acceleration of a fluid particle becomes

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}. \tag{12}$$

Note: The Lagrangian derivative (i.e. following fluid particles) is given in terms of <u>Eulerian</u> (i.e. fixed point) measurements. It is vital to understand exactly how to compute expressions like $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ for a given velocity.

Example 1.5.2 (advective derivative) *Consider an accelerating fluid flow, such as the log flowing through a narrowing channel. Suppose*

$$\mathbf{u} = (U + kx, U - ky, 0).$$

Then

- The flow is steady since $\dfrac{\partial \mathbf{u}}{\partial t} = \mathbf{0}$.
- The advective term is

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \left((U + kx)\frac{\partial}{\partial x} + (U - ky)\frac{\partial}{\partial y} \right) (U + kx, U - ky, 0).$$

  Hence the acceleration of the log is

$$\frac{D\mathbf{u}}{Dt} = (k(U + kx), -k(U - ky), 0)\,.$$

1.6 Mass conservation

One of the fundamental laws of continuum mechanics is the law of mass conservation. That is, fluid is neither created or destroyed.

Consider an arbitrary finite volume, V, which is fixed in a fixed frame of reference. V is bounded by the surface S and $\mathbf{n}$ represents a unit normal on S outward from V.

A fluid occupies the space of which V is a subset. The fluid has velocity $\mathbf{u}(\mathbf{r}, t)$ and density $\rho(\mathbf{r}, t)$. Fluid can flow in and out of V, and its density can change (within V).

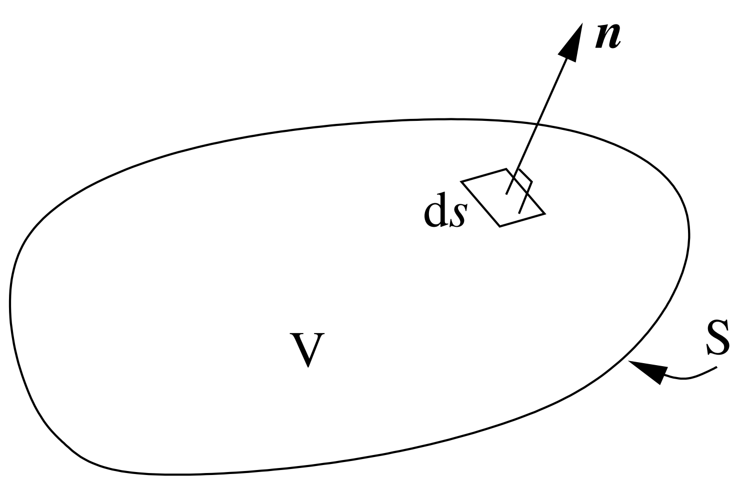


Figure 8: The mass inside a volume V can change only because because mass flows in or out through the surface S.

The mass contained in V is

$$\int_V \rho(\mathbf{r}, t) dV.$$

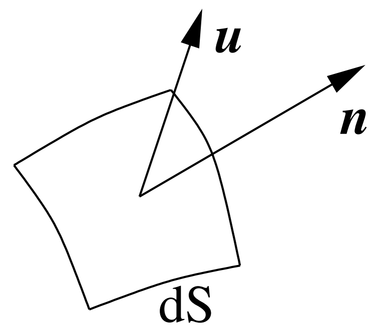


Figure 9: The flux through a surface element is determined by the inner product $\mathbf{u} \cdot \mathbf{n}$.

The flux of mass *out of* V ($\mathbf{n}$ points outward) is calculated using the following observations, see Fig. 9. In time dt the fluid on the surface element dS is transported a distance $|\mathbf{u}|dt$ in the direction of $\mathbf{u}$. Projected onto the direction $\mathbf{n}$ perpendicular to S, this corresponds to a distance $\mathbf{u} \cdot \mathbf{n}dt$, and in turn to a volume $(\mathbf{u} \cdot \mathbf{n})dtdS$. Then the total mass flowing in or out of dS is $\rho(\mathbf{u} \cdot \mathbf{n})dtdS$, and the rate of mass transport is this quantity divided by dt. This means the total flux of mass out of V is

$$\int_S \rho \mathbf{u} \cdot \mathbf{n} dS.$$

Definition 1.6.1 (mass flux density) *The quantity* $\mathbf{j} = \rho \mathbf{u}$ *is called the mass flux density.*

Now the rate of change of mass in V must equal the rate of change of mass in/out of V through S, which means that

$$\frac{\partial}{\partial t} \int_V \rho(\mathbf{r}, t) dV = -\int_S \mathbf{j} \cdot \mathbf{n} dS \tag{13}$$

Since V does not change with t and using the divergence theorem for the right hand side of (13) we get

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \boldsymbol{\nabla} \cdot \mathbf{j} dV$$

or

$$\int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} \right) dV = 0.$$

This is true for <u>any</u> fixed V, so we must have

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{14}$$

at every point in the fluid. This is called the <u>mass conservation equation</u> or the <u>continuity equation</u>.

Remark 1.6.2 (conservation laws) *The structure of (14) is very general, and applies to any conserved quantity; an equation of the form (14) is also called a conservation law. Its basic ingredients are a conserved density (e.g. mass density or energy density), and a flux (i.e. a mass flux or energy flux).*

1.7 Incompressibility

Definition 1.7.1 (Incompressibility) *A fluid is said to be* <u>*incompressible*</u> *if the density of each fluid 'particle' is constant, (i.e.* $\frac{D\rho}{Dt} = 0$*).*

From (14) we have

$$0 = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{u}$$

So ($\rho > 0$) an incompressible fluid satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{15}$$

No fluid is completely incompressible, but even gases are often sufficiently incompressible for (15) to apply to their motion. Incompressibility is a valid approximation if

- The flow speed is much less than the velocity of sound;
- Timescales are much larger than (sound frequency)$^{-1}$.

Of course this leaves out everything to do with sound waves, which are due entirely to compressible effects.

Remark 1.7.2 (Constant density) *A particular case of an incompressible fluid is one in which $\rho =$ const everywhere in space and time, and hence $\frac{D\rho}{Dt} = 0$. However, this need not be the case. For example, air in the atmosphere has different temperatures, resulting in different densities; as a result, hot air will rise. But in a frame moving with the air, the density may still be the same, and incompressibility satisfied.*

1.8 Streamfunction for two-dimensional, incompressible flows

In most circumstances, the incompressibility condition (15) is awkward to satisfy. Some progress can be made by using the following

Theorem 1.8.1 (Vector potential) *If $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, then there exists a* <u>*vector field*</u> $\mathbf{A}(\mathbf{r}, t)$ *s.t.*

$$\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{16}$$

The vector field $\mathbf{A}$ is called the *vector potential.* If $\mathbf{u}$ is represented like that, $\mathbf{u}$ is clearly incompressible. However, $\mathbf{A}$ is far from unique: if the gradient of any scalar function is added to it, the same $\mathbf{u}$ results. Thus to find a unique $\mathbf{A}$, constraints must be applied to it, and we are back to square one! However, the situation is different if the flow is two-dimensional. In that case, $\mathbf{A}$ must point in the direction perpendicular to the plane, and we can write $\mathbf{A} = \psi(x, y, t)\hat{\mathbf{z}}$, where ψ is a single scalar quantity. We will consider the case of a Cartesian and of a polar coordinate system in the plane separately.

Cartesian coordinates : Here $\mathbf{A} = \psi(x, y, t)\hat{\mathbf{z}}$, so ψ is written as a function of the two Cartesian coordinates x, y in the plane. A single scalar function ψ corresponds exactly to the expected number of degrees of freedom of the flow: there are two components of the velocity, and one constraint $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Then

$$\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right),$$

which determines how the velocity field is determined in terms of the stream function. In other words, we have represented the components u, v of a two-dimensional velocity field as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{17}$$

By construction, this representation guarantees that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, as can be checked directly.

In addition, ψ has a very convenient and useful physical interpretation. Let $(x(s), y(s)$ be a streamline, parameterized by s. Then the change of ψ along the streamline is

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial \psi}{\partial x}\lambda(s)u + \frac{\partial \psi}{\partial y}\lambda(s)v = \lambda(s)\left(-vu + uv\right) = 0,$$

having used (8) in the first step, and (17) in the second. It follows that $\psi(x, y, t) = const$ along a streamline of the flow. This motivates the following

Definition 1.8.2 (Streamfunction) *We call the function $\psi(x, y, t)$ which is constant along streamlines the streamfunction of the flow.*

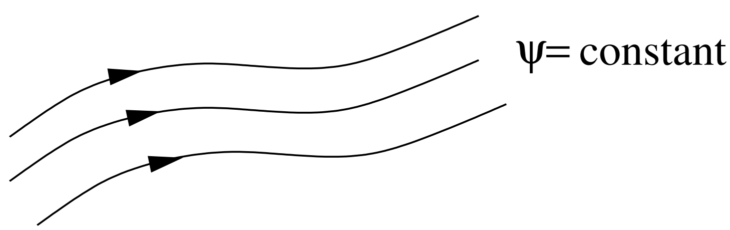


Figure 10: The streamfunction is constant along streamlines of the flow.

Note: For steady flows, the streamlines do not cross each other and fluid does not cross the streamlines.

Example 1.8.3 (Vortex) *Consider the following flow:*

$$\mathbf{u} = \frac{a}{x^2 + y^2}(y, -x) \equiv (u, v). \tag{18}$$

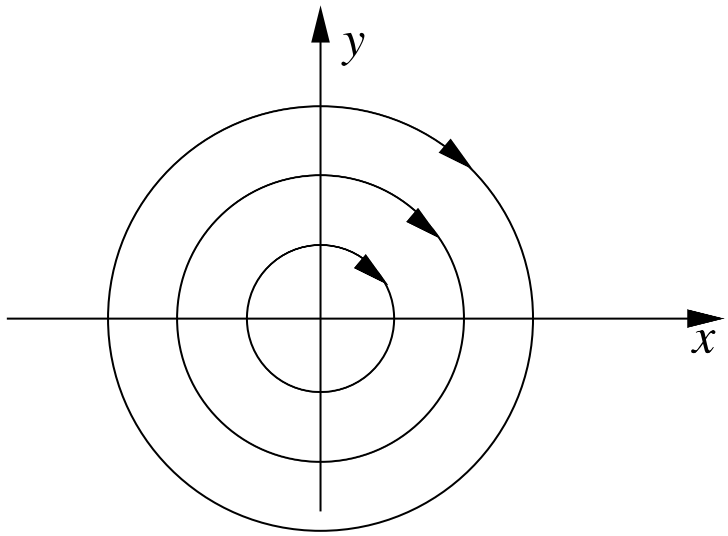


Figure 11: Streamlines of the vortex (18)

First we must confirm that this flow is incompressible, i.e. that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Now

$$\frac{\partial u}{\partial x} = -\frac{2axy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{2ayx}{(x^2+y^2)^2},$$

which indeed add up to zero.

The streamfunction must satisfy

$$\frac{\partial \psi}{\partial y} = \frac{ay}{x^2+y^2}, \quad \frac{\partial \psi}{\partial x} = \frac{ax}{x^2+y^2}.$$

From the first equation, $\psi = a\ln(x^2+y^2)/2 + f(x)$, and then from the second equation $f'(x) = 0$. Thus the streamfunction is

$$\psi = \frac{a}{2}\ln\left(x^2+y^2\right), \tag{19}$$

only defined up to a constant, of course. Clearly, the streamlines $\psi = const$ are circles, as expected.

Polar coordinates : Clearly, a streamfunction such as (19) is represented more easily in polar coordinates. In that case, we have $\mathbf{A} = \psi(r,\theta,t)\hat{\mathbf{z}}$. Using the formula for the curl in cylindrical polars, we find

$$\mathbf{u} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\mathbf{r}} - \frac{\partial \psi}{\partial r}\hat{\boldsymbol{\theta}},$$

so that the two velocity components are represented as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}. \tag{20}$$

From the geometrical interpretation of ψ it is clear that streamlines are given by $\psi(r,\theta) = const$ (of course this can also be checked by direct calculation). Thus

$$\psi(r,\theta,t) = \psi_C(r\cos\theta, r\sin\theta, t),$$

where we have written ψ_C for the streamfunction in Cartesians.

Example 1.8.4 (Vortex in polar coordinates) *From the above it follows that the streamfunction (19) of Example 1.8.3 can be written in polar coordinates as*

$$\psi = \frac{a}{2}\ln\left(x^2+y^2\right) = a\ln r.$$

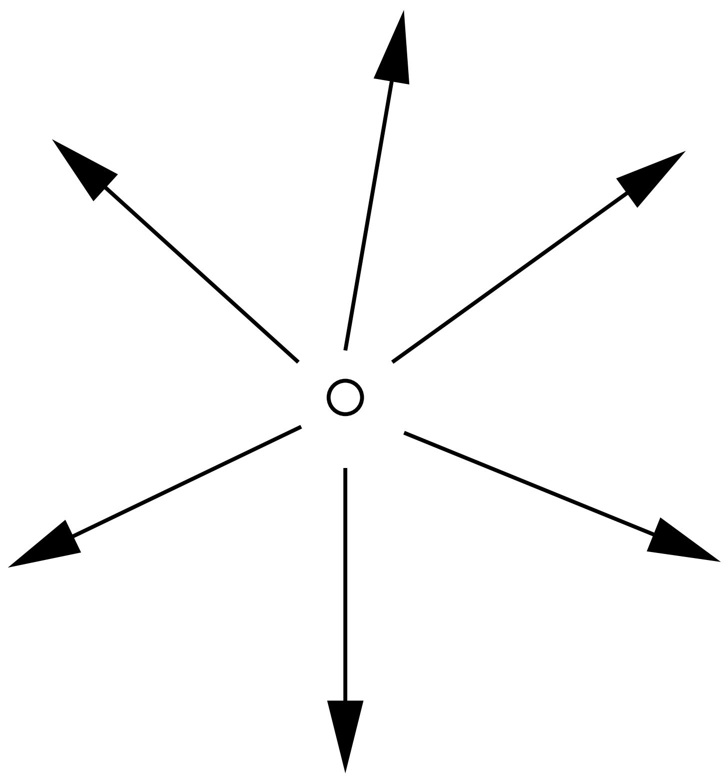


Figure 12: Streamlines of a source.

Example 1.8.5 (Source in polar coordinates) *In the case of a source, the flow is purely radial, see Fig. 12, so that $u_\theta = 0$ and u_r is independent of θ.*

From (20) we have

$$\left.\begin{aligned} u_r &= f(r) = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ u_\theta &= 0 = -\frac{\partial\psi}{\partial r}. \end{aligned}\right\}$$

Thus we must have $\psi = \psi(\theta)$ but $\partial\psi/\partial\theta$ independent of θ. So $\psi = A\theta$, where A is a constant. It follows that $f(r) = A/r$.

Definition 1.8.6 (Source strength) *The source strength is the flux of fluid from the source point.*

The flow is incompressible, so the flux at the origin equals the flux through any closed boundary surrounding the origin. Found, most conveniently, by measuring the flux through a circle radius r centred at the origin. The source strength is

$$m = \oint_{\text{circle}} \mathbf{u} \cdot \mathbf{n} ds = \int_0^{2\pi} \mathbf{u} \cdot \mathbf{n} r d\theta = \int_0^{2\pi} \frac{A}{r} r d\theta = 2\pi A,$$

where $\mathbf{n} = \hat{\mathbf{r}}$ in polars.

This means the mass flow through a circle surrounding the origin is independent of the radius of the circle, and equals the source of mass flux at the origin. In fact, one can show that the mass flux through *any* closed surface is m, using Green's theorem (how exactly?). In conclusion, the two-dimensional velocity field of a source is

$$\mathbf{u} = \frac{m}{2\pi r} \hat{\mathbf{r}}, \tag{21}$$

and the stream function is

$$\psi = \frac{m\theta}{2\pi}. \tag{22}$$

The velocity fields and streamfunctions derived here will be summarized in Appendix D. **Note**: In Cartesians,

$$\psi = \frac{m}{2\pi} \arctan(y/x).$$

Another take on the above statement is provided by the following interesting property of the streamfunction:

Theorem 1.8.7 (Flux across a line) *Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be two points in the plane, and ψ the streamfunction. Then the volume flux crossing any curve from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by $\psi(\mathbf{r}_1) - \psi(\mathbf{r}_0)$.*

Proof: The volume flux Q across the curve C joining the points $\mathbf{r}_0$ and $\mathbf{r}_1$ is

$$Q = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s,$$

where $\mathrm{d}s$ is the arc length along the curve. We choose $\mathbf{n}$ in the mathematically positive, counterclockwise direction. Since the tangent vector in the direction of the curve is $\mathbf{t} = (\mathrm{d}x, \mathrm{d}y)/\mathrm{d}s$, this is the orthogonal unit vector $\mathbf{n} = (\mathrm{d}y, -\mathrm{d}x)/\mathrm{d}s$, pointing in the counterclockwise direction, going from $\mathbf{r}_0$ to $\mathbf{r}_1$. Now

$$Q = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \cdot \frac{(\mathrm{d}y, -\mathrm{d}x)}{\mathrm{d}s} \mathrm{d}s = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \frac{\partial \psi}{\partial y} \mathrm{d}y + \frac{\partial \psi}{\partial x} \mathrm{d}x = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathrm{d}\psi = \psi(\mathbf{r}_1) - \psi(\mathbf{r}_0),$$

as claimed. In particular, if $\mathbf{r}_0$ and $\mathbf{r}_1$ lie on the same streamline, the flux across the streamline vanishes, which is obvious. Applied to the source (21), choose $\mathbf{r}_0 = (r, 0)$ and $\mathbf{r}_1 = (r, 2\pi)$ in polar coordinates, so that $\psi(\mathbf{r}_1) - \psi(\mathbf{r}_0) = m$, according to (22), which is indeed the source strength.

2 Flow dynamics for an incompressible inviscid flow

2.1 Forces on a fluid

Fluids move in response to the forces that act on each fluid particle. In order to apply Newton's second law in straightforward manner, we need to define a fluid "particle" (or fluid element) in such a way that the mass is constant. We also want it to be big enough (encompassing a large number of molecues), such that continuum theory applies. This can be acccomplished for example by marking each molecule in an initial lump of fluid (for example a cube). At later times, we think of the fluid particle as the volume δV containing the original molecules. As the molecules move about, the fluid volume will deform, but its mass remains constant.

There are two types of forces on a fluid element:

(i) *Body forces* are forces on a each parcel throughout the bulk of the fluid. In a gravitational potential, the force is proportional to the volume δV of the fluid element. In the typical case of a uniform gravitational field (as near the surface of the earth) the force is $\delta \mathbf{F}_b = \rho \delta V \mathbf{g}$. We will denote the *force density* (per unit volume) by $\mathbf{f}(\mathbf{r})$.

(ii) *Surface forces* are transmitted across a surface element dS of a fluid parcel, so the force is exerted *by* the fluid on the exterior of δV or vice versa.

When the fluid is a rest, the surface force must be in the direction of the normal, $\mathbf{n}$, since by definition a fluid cannot sustain any shear (it would flow). When a fluid is in motion, tangential components of the force on dS can occur. These are associated with viscosity which describes the effect that one layer of fluid in motion has on an adjacent layer.

Definition 2.1.1 (Inviscid fluid) *A fluid is said to be inviscid (also known as an ideal fluid), when surface forces act in the normal direction only.*

For an inviscid fluid, the 'surface stress' (i.e. Force per unit area) is in the direction $\mathbf{n}$ normal to an infinitesimal surface element dS *even when the fluid is in motion*. In other words, the force exerted by the exterior fluid on any surface element dS of the fluid element δV is $d\mathbf{F}_s = -p\mathbf{n}dS$, where $p(\mathbf{r}, t)$ is the pressure. The pressure has units of force per unit area. It is directed inwards because fluids are usually in a state of compression.

2.2 Equation of motion

We apply Newton's law to a fluid element with volume δV and position $\mathbf{r}(\mathbf{a}, t)$ as defined above. The mass δm of this fluid element is constant by construction. Then Newton's equation of motion is

$$\delta m \frac{D\mathbf{u}}{Dt} = \delta \mathbf{F}, \tag{23}$$

where $\delta \mathbf{F}$ is the total force acting on the fluid particle. According to the above, $\delta \mathbf{F}$ is composed of surface and body forces:

$$\delta \mathbf{F} = \delta \mathbf{F}_b + \delta \mathbf{F}_s = \mathbf{f}(\mathbf{r}, t)\delta V - \int_{\delta S} p\mathbf{n}dS,$$

where the integral is over the surface of the fluid element. Using Gauss' theorem, the surface integral can be written as

$$\int_{\delta S} p\mathbf{n}dS = \int_{\delta V} \boldsymbol{\nabla} p dV \approx \boldsymbol{\nabla} p \delta V,$$

if the fluid particle is sufficently small so $\boldsymbol{\nabla} p$ can be treated as constant over its volume.

Thus $\delta\mathbf{F} = (\mathbf{f}(\mathbf{r},t) - \boldsymbol{\nabla} p)\delta V$, and since we also have $\delta m = \rho(\mathbf{r},t)\delta V$, (23) becomes

$$\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla} p + \mathbf{f}. \tag{24}$$

Using the definition of the convective derivative (12), this can be written explicitly as

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = -\boldsymbol{\nabla} p + \mathbf{f}. \tag{25}$$

Either (25) or (24) are known as Euler's equation (due to Euler 1756), on which the rest of this course will be based. Notice that in deriving (25), we have made no assumption about the fluid being incompressible.

What at first seems strange about Euler's equation is that the pressure p on the right is not specified. Indeed, a casual count shows that (25) is a set of three equations for 4 unknowns (three components of $\mathbf{u}$ and the pressure. We need another equation! In this course, this will be the incompressibility condition (15): $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$. Together with (25), we then have 4 equations for 4 unknowns.

A good analogy for this situation comes from point mechanics: Euler's equation (25) plays the role of Newton's equation $m\mathbf{a} = \mathbf{F}$. However, only the body forces $\mathbf{f}$ (such as gravity) are specified explicitely, p is unknown. The reason is that p plays the role of a *constraint force*, with the constraint being $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$. We have to find p, such that the constraint is satisfied at all times. Just as in mechanics, solving the constraint is the most difficult bit. While Lagrangian mechanics teaches us how to solve the problem by choosing an appropriate coordinate system, we don't have this luxury in fluid mechanics: it is not possible in general to find a representation of the velocity field $\mathbf{u}$ with two independent components, which satisfies the constraint identically. The only exception, as we have seen, is the case of two-dimensional flows, for which we have the stream function representation.

2.3 The momentum flux

In Section 1.6 we derived the law for the conservation of mass. In fact, Euler's equation (24) is the statement of another conservation law, that of momentum conservation. To obtain the momentum, we multiply the velocity $\mathbf{u}$ by the density ρ, to find the momentum density per unit volume $\rho\mathbf{u}$. To be able to write conservation of momentum in the form of the continuity equation (14), we write the equation for each Cartesian component separately:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial M_{ij}}{\partial x_j} = f_i. \tag{26}$$

On the right-hand-side is the i'th component force density $\mathbf{f}$, since according to Newton's law the time derivative of the momentum is the force. Indeed, if we define the flux of x-momentum p_x by

$$f_j^{(p_x)} = M_{1,j},$$

and there are no external forces, then it follows from (26) that

$$\frac{\partial p_x}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{f}^{(p_x)} = 0. \tag{27}$$

This describes the conservation of x-momentum, and has exactly the same form as (14).

We now want to show that (26) is equivalent to (25), with the momentum flux tensor

$$M_{ij} = \rho u_i u_j + p\delta_{ij}. \tag{28}$$

Indeed, inserting (28) into (26), we have

$$\frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t} + \frac{\partial \rho}{\partial x_j} u_i u_j + \rho \frac{\partial u_i}{\partial x_j} u_j + \rho u_i \frac{\partial u_j}{\partial x_j} + \frac{\partial p}{\partial x_j}\delta_{ij} =$$
$$-\left(\frac{\partial \rho}{\partial x_j} u_j + \rho \frac{\partial u_j}{\partial x_j}\right) u_i + \rho \frac{\partial u_i}{\partial t} + \frac{\partial \rho}{\partial x_j} u_i u_j + \rho \frac{\partial u_i}{\partial x_j} u_j + \rho u_i \frac{\partial u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} =$$
$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} = f_i,$$

which is precisely the Euler equation (25); in the second step we have used the continuity equation (14).

For *steady flows* it is often useful to consider the momentum balance over a control volume V. Since the flow is steady, the left hand side of (26) is zero. We also assume that $\mathbf{f} = -\rho\boldsymbol{\nabla}\Phi$, and that ρ is constant. Then

$$0 = \int_V \frac{\partial M_{ij}}{\partial x_j} dV + \rho \int_V \frac{\partial \Phi}{\partial x_i} dV = \int_S (M_{ij} n_j + \rho\Phi n_i)\, dS = \int_S (u_i u_j n_j + (p + \rho\Phi) n_i)\, dS$$

by the divergence theorem. So, back in vector form,

$$\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + (p + \rho\Phi)\mathbf{n} dS = 0 \tag{29}$$

for any closed surface S inside or bounding the fluid. This is the momentum integral theorem, of which we will make lots of use!

We now give a few simple examples of flows which are solutions of (25) in the absence of an external force $\mathbf{f} = 0$. In each case, it is crucial to also confirm that the incompressibility condition $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ is satisfied. Expressions for $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ in various coordinate systems are provided in chapter 0. We assume there is no external force.

Example 2.3.1 (Uniform flow) *We have* $\mathbf{u} = \mathbf{U}$. *This must be a solution, as each particle travels along straight lines, and thus no forces are involved.*

Obviously, $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, so the flow is incompressible. Turning to (25), the flow is steady, $\frac{\partial \mathbf{u}}{\partial t}$ and $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = 0$, since $\mathbf{u}$ is constant. Thus $\boldsymbol{\nabla} p = 0$, and it follows that $p = const$, as expected.

Example 2.3.2 (Source in three dimensions) *The flow field must point radially outward from the origin, and thus* $\mathbf{u} = f(r)\hat{\mathbf{r}}$. *Then in spherical coordinates,*

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2 f(r))}{\partial r}.$$

For this to vanish, we must clearly have $f(r) = A/r^2$. The total flux from the origin must equal the flux through a sphere of radius R. Since the outward normal is $\mathbf{n} = \hat{\mathbf{r}}$, we have

$$m = \int_{S_R} \mathbf{u} \cdot \mathbf{n} dS = 4\pi R^2 f(R) = 4\pi A.$$

Thus the incompressible velocity field corresponding to a source of strength m is

$$\mathbf{u} = \frac{m}{4\pi r^2}\hat{\mathbf{r}}. \tag{30}$$

Now we test for the solution of Euler's equation. Again the flow is steady, and in a spherically symmetric situation Euler's equation becomes

$$\rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \rho u_r \frac{\partial u_r}{\partial r}\hat{\mathbf{r}} = -\frac{m^2 \rho}{8\pi^2 r^5}\hat{\mathbf{r}} = -\boldsymbol{\nabla} p.$$

Thus the pressure only depends on the radial coordinate, with

$$\frac{\partial p}{\partial r} = \frac{m^2 \rho}{8\pi^2 r^5},$$

and so

$$p = p_0 - \frac{m^2 \rho}{32\pi^2 r^4}.$$

Thus with increasing r, the pressure increases. This is expected, as the flow slows down in the radial direction, which means a force must be holding it back.

2.4 Hydrostatics

For a fluid at rest, $\mathbf{u} = 0$ and so it follows from (24) that

$$\boldsymbol{\nabla} p = \mathbf{f}.$$

In the case of a graviational potential (which is conservative), the force is proportional to the mass, mulitplied by the gradient of the potential $\boldsymbol{\nabla}\Phi$. But since $\mathbf{f}$ is the force density per unit volume, we obtain $\mathbf{f} = -\rho\boldsymbol{\nabla}\Phi$. Then, if $\rho = const$, we obtain $\boldsymbol{\nabla}(p + \rho\Phi) = 0$. Integrating, this leads to

$$p + \rho\Phi = 0. \tag{31}$$

In the case of a constant gravitational force, $\mathbf{f} = -\rho g\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is pointing vertically upwards. The corresponding potential is $\Phi = gz$, so that

$$p + \rho g z = const, \qquad \text{in the fluid.} \tag{32}$$

In the case that $\rho = \rho(z)$ (as is relevant for oceans or atmospheres),

$$\boldsymbol{\nabla} p = -\rho(z) g\hat{\mathbf{z}}, \qquad \Rightarrow \qquad p(z) = p(z_0) - g\int_{z_0}^{z} \rho(z')dz'.$$

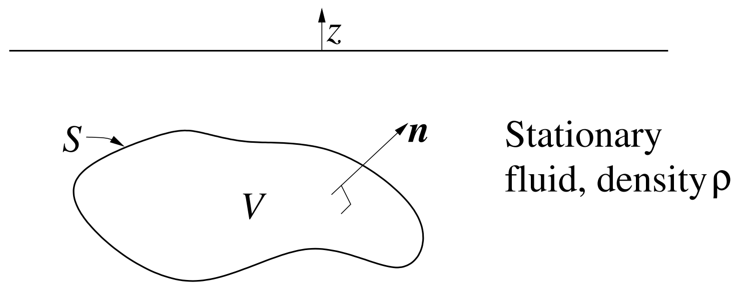


Figure 13: The force on a submerged body equals the weight of the displaced fluid.

Example 2.4.1 (Archimedes' principle (~ 250BC)) *Let us apply these ideas to a submerged body.*

$$\begin{aligned}
\textit{Force on submerged body} &= -\int_S p\mathbf{n}dS \\
&= -\int_V \boldsymbol{\nabla} p dV, \qquad \textit{Divergence Theorem} \\
&= \int_V \rho g \hat{\mathbf{z}} dV, \qquad \hat{\mathbf{z}} \textit{ unit normal in } z\textit{-direction} \\
&= \rho g V \hat{\mathbf{z}}
\end{aligned}$$

Therefore Archimedes' principle becomes: force on body = weight of fluid displaced is $\rho g V$.

A body will float if its total weight is smaller than that of the fluid it displaces. According to an anecdote, Archimedes used this principle to determine if a crown was made of pure gold.

Figure 14: Left: the crown and a gold nugget used for reference have the same weight. Right: The crown of lower density receives a greater upward force, so it becomes lighter when submerged.

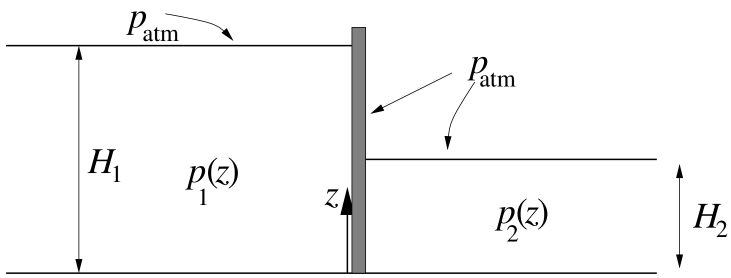


Figure 15: Forces act on both sides of a lock gate; everything is per unit length (into the page).

Example 2.4.2 (Lock gates) *Consider water on both sides of a lock gate.*

To the left of the gate, we can use (32), and the atmosphere surrounding the fluid surface is at constant pressure p_{atm}. Since the pressure must be continuous across the interface (we will *derive* this condition more formally in the next section), we also have $p = p_{atm}$ on $z = H_1$, just inside the fluid. It follows that

$$p_1(z) = p_{atm} + \rho g(H_1 - z), \qquad 0 < z < H_1.$$

Similarly, to the right of the gate,

$$p_2(z) = \begin{cases} p_{atm}, & z > H_2, \\ p_{atm} + \rho g(H_2 - z), & 0 < z < H_2, \end{cases}$$

Thus the force exerted by the fluid on the gate (per unit width of the gate) is

$$\mathbf{F}_1 = \int_0^{H_1} p_1(z)\mathbf{n} dz, \tag{33}$$

where $\mathbf{n} = \hat{\mathbf{x}}$ is the vector pointing out of the fluid toward the gate. Thus the x-component of the force from the left is

$$F_1 = \mathbf{F}_1 \cdot \hat{\mathbf{x}} = \int_0^{H_1} p_1 dz = p_{atm} H_1 + \tfrac{1}{2}\rho g H_1^2,$$

and the x-component of the force (per unit width) exerted by the fluid on the gate from the right is

$$F_2 = -\int_0^{H_1} p_2 dz = -p_{atm} H_1 - \tfrac{1}{2}\rho g H_2^2,$$

where now $\mathbf{n} = -\hat{\mathbf{x}}$, and hence the - sign. Thus the net force in the x-direction is

$$F = F_1 + F_2 = \frac{1}{2}\rho g(H_1^2 - H_2^2),$$

which is toward the right ($H_1 > H_2$). In one of the problems, we will calculate the torque on the gate.

2.5 Kinematic boundary condition

As with any PDE, it is important to understand the boundary conditions Euler's equation has to satisfy. The kinematic boundary condition expresses the conditions on the motion of the fluid at the boundary. We will deal with solid boundaries first, and then generalize to moving boundaries, for example those between two fluids or between fluid and air.

2.5.1 Solid boundary

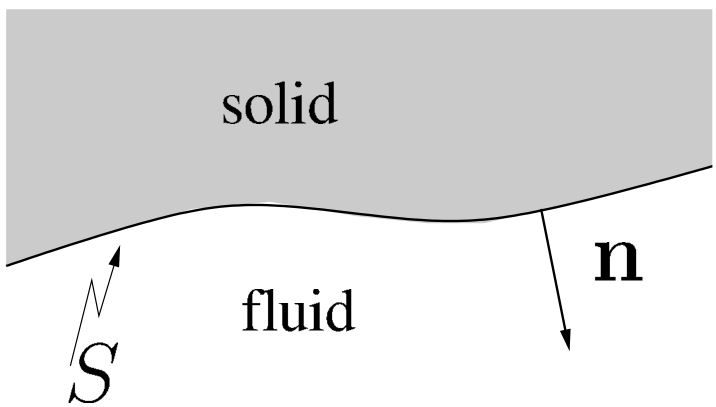


Figure 16: A solid boundary with normal pointing into the fluid.

Let us consider the flow near a solid surface, whith normal $\mathbf{n}$ pointing into the fluid. We begin by assuming that the solid be at rest. At any point on the surface, it is clear that the fluid cannot penetrate the solid. This means that the normal component of the velocity must be zero on the surface: $\mathbf{u}(\mathbf{r}, t) \cdot \mathbf{n} = 0$ for any $\mathbf{r} \in S$. Instead, the flow can only have a component parallel to the solid.

Now let us take the slightly more general situation that the solid is moving. At any point on the surface, we can go into a frame of reference moving with the solid, such that the point is at rest in this coordinate system. This leads us to the requirement

$$\left(\mathbf{u}(\mathbf{r}, t) - \mathbf{u}^{(solid)}(\mathbf{r}, t)\right) \cdot \mathbf{n} = 0 \quad \text{for } \mathbf{r} \in S, \tag{34}$$

which is the boundary condition to be satisfied at solid surfaces.

Example 2.5.1 (A solid wall at rest) *Let the wall occupy the plane $z = 0$, and the fluid be at $z > 0$. Then $\mathbf{n} = \hat{\mathbf{z}}$, and the boundary condition is $\mathbf{u} \cdot \hat{\mathbf{z}} = 0$, or $u_z = 0$ for $z = 0$.*

Note that the other components of the velocity field, u_x and u_y, are arbitrary! The fluid can "slip" over the solid without resistance, which is consistent with out notion of a frictionless or ideal fluid.

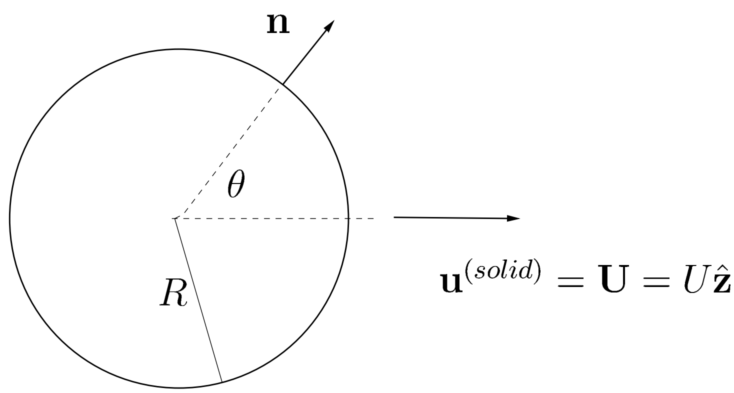


Figure 17: A solid boundary with normal pointing into the fluid.

Example 2.5.2 (A sphere of radius R moving through a fluid at velocity $\mathbf{u}$) *According to (34),* $(\mathbf{u}-\mathbf{u}^{(solid)})\cdot\mathbf{n}=0$ *on the surface of the sphere.*

At a given instant, let the centre of the sphere be at the centre of a spherical polar coordinate system, and choose the z-axis such that $\mathbf{u}^{(solid)}=U\hat{\mathbf{z}}$. Then $\mathbf{n}=\hat{\mathbf{r}}$, so that $\mathbf{u}\cdot\mathbf{n}=u_r$, and $\mathbf{u}^{(solid)}\cdot\mathbf{n}=U(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})=U\cos\theta$. Thus the boundary condition reads

$$u_r = U\cos\theta, \quad \text{for } r=R.$$

2.5.2 Moving boundary

Now we formulate a generalization of the boundary condition of the preceeding section, in which the boundary is allowed to deform in any way, for example the surface of the ocean. To this end we describe the (time-dependent) boundary by the level line of some function: let $S(\mathbf{r},t)=0$ describe the equation of a surface between two fluids. As the flow evolves, particles remain on the surface S if

$$\frac{DS}{Dt}=\frac{\partial S}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}S=0 \tag{35}$$

Namely, we want that a Lagrangian particle $\mathbf{r}(\mathbf{a},t)$, once on the surface, remains on surface. This means that

$$0=\frac{d}{dt}S(\mathbf{r}(\mathbf{a},t),t)=\frac{\partial S}{\partial t}+\dot{\mathbf{r}}\cdot\boldsymbol{\nabla}S=\frac{\partial S}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}S,$$

as claimed.

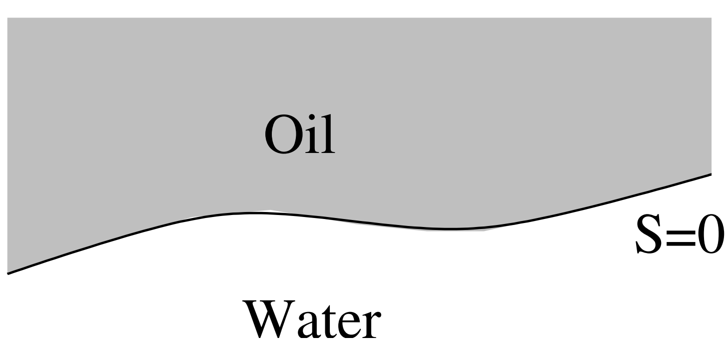


Figure 18: The interface between a phase of oil and a phase of water is given by the level curve $S(\mathbf{r},t)=0$.

Let us apply this to the interface between two liquids, as shown in Fig. 18. Then

$$\left.\begin{aligned} \frac{DS}{Dt} &= \frac{\partial S}{\partial t} + \mathbf{u}^{(oil)} \cdot \boldsymbol{\nabla} S = 0, \qquad \text{on } S = 0 \text{ from above} \\ \frac{DS}{Dt} &= \frac{\partial S}{\partial t} + \mathbf{u}^{(water)} \cdot \boldsymbol{\nabla} S = 0, \qquad \text{on } S = 0 \text{ from below} \end{aligned}\right\}$$

Now $\boldsymbol{\nabla} S$ is normal to the surface $S = const$, so $\mathbf{n} = \boldsymbol{\nabla} S/|\boldsymbol{\nabla} S|$ is the unit normal to the surface $S = 0$. It follows that

$$\mathbf{u}^{(oil)} \cdot \mathbf{n} = \mathbf{u}^{(water)} \cdot \mathbf{n}. \qquad \text{on } S = 0 \tag{36}$$

This expresses the intuitive notion that the normal component of the velocity on either side of the interface must be equal.

Note that (36) expresses exactly the same thing as (34) for a solid boundary, except that the velocity of the solid is now prescribed (and can only be a translation or rotation).

Example 2.5.3 (A collapsing/expanding bubble) *An example we will consider in section 3.4 below, is the collapse of a spherical bubble inside a fluid. Let us assume that the radius $R(t)$ of the bubble changes according to some time program, which we will calculate later on. If we choose*

$$S(\mathbf{r}, t) = r - R(t), \tag{37}$$

$S(\mathbf{r}, t) = 0$ *indeed describes the surface of the bubble.*

Now computing (35), we find

$$0 = \frac{\partial S}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} S = -\dot{R} + \mathbf{u} \cdot \hat{\mathbf{r}} \frac{\partial S}{\partial r} = -\dot{R} + u_r,$$

where $u_r = \mathbf{u} \cdot \hat{\mathbf{r}}$ is the radial compoment of the velocity in spherical coordinates. In other words, the boundary on the surface of the bubble is

$$u_r = \dot{R}, \tag{38}$$

which is very intuitive, and could have been written down without any calculation!

2.6 Dynamic boundary condition

To derive a boundary condition the pressure has to satisfy on the boundary of a free surface between two fluids (e.g. interface between oil/water) we use momentum conservation (26).

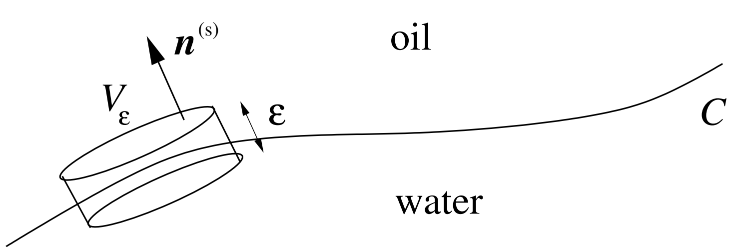


Figure 19: A "pillbox" argument at the interface between two phases.

Let us integrate (26) throughout a small disc V_ϵ of thickness ϵ and let $\epsilon \to 0$ (cf. Fig. 19). We only consider the direction $\mathbf{n}^{(s)}$ normal to the interface, in which $\mathbf{u} \cdot \mathbf{n}^{(s)}$ is continuous. Then

$$\int_{V_\epsilon} \frac{\partial}{\partial t}(\rho \mathbf{u} \cdot \mathbf{n}^{(s)}) dV \to 0, \qquad \text{as } \epsilon \to 0,$$

and therefore

$$0 = \lim_{\epsilon \to 0} \int_{V_\epsilon} n_i^{(s)} \frac{\partial}{\partial x_j} M_{ij} dV = \lim_{\epsilon \to 0} \int_{S_\epsilon} n_i^{(s)} M_{ij} n_j dS = \left[n_i^{(s)} M_{ij} n_j^{(s)} \right]_+ - \left[n_i^{(s)} M_{ij} n_j^{(s)} \right]_- .$$

But this means that

$$\left[\rho (\mathbf{u} \cdot \mathbf{n}^{(s)})^2 + p \mathbf{u} \cdot \mathbf{n}^{(s)} \right]_+ - \left[\rho (\mathbf{u} \cdot \mathbf{n}^{(s)})^2 + p \mathbf{u} \cdot \mathbf{n}^{(s)} \right]_- = 0. \tag{39}$$

Since $\mathbf{u} \cdot \mathbf{n}^{(s)}$ is continuous across the interface, we are left with

$$p_+ = p_-, \tag{40}$$

which says that the pressure is continuous across an interface. This is very intuitive, and in fact we have already used (40) in example 2.4.2. In the atmosphere, the pressure has the constant value p_{atm}, so (40) says that the fluid pressure at a liquid-air interface must be $p = p_{atm}$.

Example 2.6.1 (Axisymmetric jet impinging on a wall) *As a first example of the application of the momentum integral theorem (29), let us consider the situation shown in Fig. 20. What is the force exerted by the flow on the wall ? Ignore the effects of gravity.*

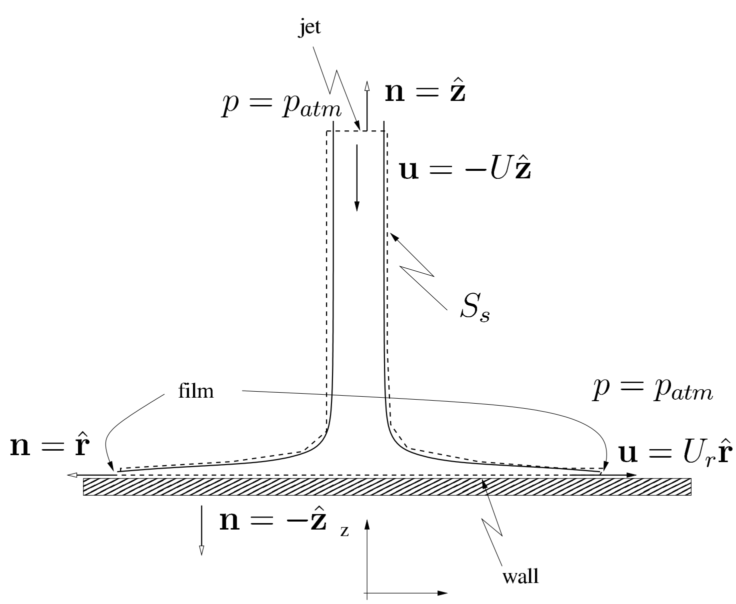


Figure 20: A jet impinging perpendicularly on a plate. The dashed line marks the control surface S.

Take a <u>control surface</u> $S = S_s + S_j + S_f + S_w$ of cylindrical shape that is composed of the free surface of the jet on one side, and the wall on the other. The surface is *closed* by a circular surface through the jet far from impact (marked "jet"), and a cylindrical surface where the fluid flows parallel to the wall (marked "film"). Assume uniform flow parallel to wall in the film region far from impact and across the cross section of the jet $\mathbf{u} = -U\hat{\mathbf{z}}$. In the preceeding section we have shown that the pressure on all boundaries exposed to the atmosphere is p_{atm}. This includes the jet itself, which has assumed the pressure p_{atm}. Then in the absence of body forces,

$$\int_{S=S_j+S_s+S_f+S_w} (\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) + p\mathbf{n})\, dS = 0. \tag{41}$$

Now

$$\int_S p\mathbf{n}dS = \int_S (p - p_{atm})\mathbf{n}dS + \int_S p_{atm}\mathbf{n}dS = -\int_{S_w} (p - p_{atm})\hat{\mathbf{z}}dS$$

since $\int_S p_{atm}\mathbf{n}dS = \int_V \boldsymbol{\nabla} p_{atm} dV = 0$ and $p = p_{atm}$ everywhere else. Also,

$$\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dS = \int_{S_j} \rho U^2\hat{\mathbf{z}}dS + \int_{S_f} \rho U_r^2\hat{\mathbf{r}}dS$$

since $\mathbf{u}\cdot\mathbf{n} = 0$ on the wall and on the surface of the liquid. Thus (41) becomes

$$\int_{S_w} (p - p_{atm})\hat{\mathbf{z}}dS = \int_{S_j} \rho U^2\hat{\mathbf{z}}dS + \int_{S_f} \rho U_r^2\hat{\mathbf{r}}dS.$$

Multiplying this by $\hat{\mathbf{z}}$ to only consider the z-component of the momentum balance one obtains

$$\int_{S_w} (p - p_{atm})dS = \rho U^2 A,$$

where A is the cross section of the jet. But then the force on the wall in the $\hat{\mathbf{z}}$-direction is

$$F_z = -\int_{S_w} (p - p_{atm})dS = -\rho U^2 A.$$

2.7 Bernoulli's equation for steady flows

The Euler equation for steady motion is

$$\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p - \rho\boldsymbol{\nabla}\Phi,$$

assuming that the body forces possess a potential Φ. We can transform the term on the right using the vector identity (see Appendix A)

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\nabla}(\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\boldsymbol{\omega} \tag{42}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ is the vorticity. Thus we obtain (taking $\rho = const$)

$$-\mathbf{u}\times\boldsymbol{\omega} = -\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2). \tag{43}$$

To make the left hand side disappear we take the dot-product with $\mathbf{u}$ on both sides to find

$$\mathbf{u}\cdot(\mathbf{u}\times\boldsymbol{\omega}) = 0 = \mathbf{u}\cdot\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2).$$

The definition of a streamline is $\dfrac{d\mathbf{r}}{ds} = \lambda(s)\mathbf{u}$, so

$$\frac{d}{ds}(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2) = \frac{dx_i}{ds}\frac{\partial}{\partial x_i}(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2) = \lambda(s)\mathbf{u}\cdot\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2) = 0.$$

Consequently, the quantity

$$H = p + \rho\Phi + \rho\mathbf{u}^2/2 = const, \qquad \text{along any streamline } \mathbf{r}(s) \text{ in the flow.} \tag{44}$$

In the problems we show that the energy flux is $\mathbf{j}_E = \mathbf{u}H$. Thus (44) can be seen as a statement of energy conservation, where the Bernoulli quantity H is conserved for a fluid element along a streamline. Note: A very nice discussion of the Bernoulli effect and of pressure forces is presented in the film: "Pressure fields and acceleration", found on Youtube.

Example 2.7.1 (A three-dimensional source) *According to Example 2.3.2, the velocity file is*

$$\mathbf{u} = \frac{m}{4\pi r^2}\hat{\mathbf{r}}.$$

Evidently, stream lines are rays pointing radially outward from the center, and

$$\frac{1}{2}\mathbf{u}^2 = \frac{m^2}{32\pi^2 r^4}.$$

According to (44),

$$p = -\frac{m^2\rho}{32\pi^2 r^4} + const,$$

where the constant could in principle be different for each streamline. However, the pressure must go to the same constant far from the source, so we reach the same conclusion as in Example 2.3.2. However, Bernoulli's theorem is particularly powerful for more complicated flows, for which simple solutions are not available, as the next example shows.

Example 2.7.2 (Flow out of a tank) *A tank of uniform cross section A_0 has a small hole, with area A_e, and at a height h above the base, see Fig. 21. The height of the fluid above the hole is H. What is the flow speed out of the drain ?*

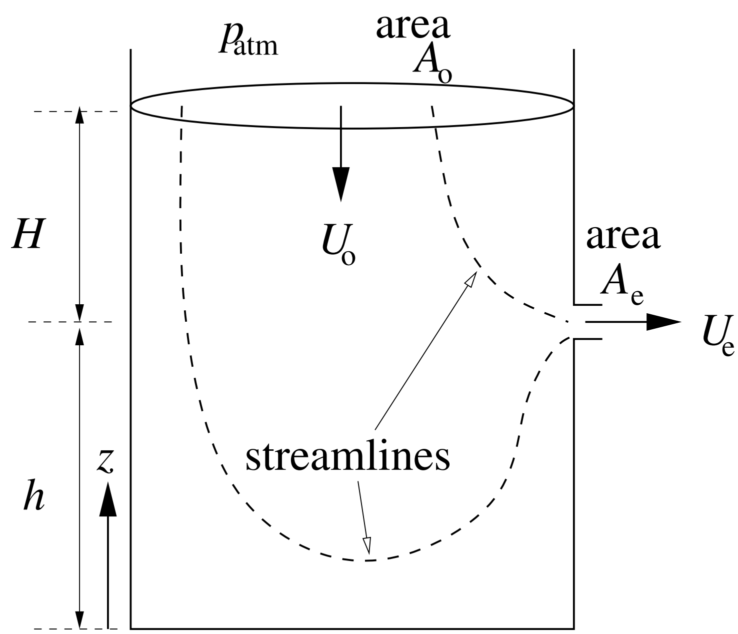


Figure 21: Water flowing out of a tank.

First, conservation of mass gives

$$\rho A_0 U_0 = \rho A_e U_e, \tag{45}$$

since the mass fluxes across two boundaries equal. Since the top fluid level decreases very slowly, the flow is approximately steady. Second, we want to apply Bernoulli's theorem to a streamline that exits the hole. It is important to notice that any such streamline must originate from the *surface* of the container (dashed lines), and ends at the hole. Namely, all streamlines must be parallel to a stationary wall, and thus cannot originate from the wall. By contrast, the top surface is moving downward (albeit slowly), and thus the velocity field has a component pointing downward, in the local direction of the streamline. The potential in (44) is $\Phi = gz$. The pressure at the top surface and just outside the hole is $p = p_{atm}$. Then

- At surface: $p_{atm} + \rho g(h + H) + \frac{1}{2}\rho U_0^2 = C$
- At exit: $p_{atm} + \rho gh + \frac{1}{2}\rho U_e^2 = C$,n

and along the same streamline the constant C is the same. Eliminating C and p_{atm} and using (45) we find

$$U_e^2[1 - (A_e/A_0)^2] = 2gH.$$

It is reasonable to assume that $A_e \ll A_0$, so

$$U_e \approx \sqrt{2gH}. \tag{46}$$

This result corresponds exactly to the free fall velocity of a mass having been dropped from a height H. The reason for this is clear: a fluid particle, having left the top surface, moves along a (perhaps complicated) path dictated by the geometry of the vessel and the incompressibility constraint $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. However, since this motion is frictionless, the shape of this path is irrelevant, and the final speed is dictated by energy conservation alone.

With the main result (46) in hand, we can address two practical questions:

1. What is the draining time t_d for an initial filling height H_0 to reach 0 ?
   The equation of motion for the surface is
$$\frac{dH}{dt} = -U_0 \approx -\frac{A_e}{A_0}\sqrt{2gH}.$$
   and so
$$t_d = \int_{H_0}^{0} \frac{dt}{dH} dH = -\frac{A_0}{A_e}\int_{H_0}^{0} \frac{dH}{\sqrt{2gH}} = \frac{A_0}{A_e}\sqrt{2H_0/g}. \tag{47}$$
2. What is the horizontal distance x travelled by the jet?
   The fluid particles making up the jet are subject to gravitational acceleration only, since the pressure along the jet has the constant value $p = p_{atm}$. Hence they obey the same equations as a projectile fired horizontally with speed $U_e = \sqrt{2gH}$. The time taken by a particle to hit the floor is $t = \sqrt{2h/g}$, and so
$$x = U_e t = \sqrt{2gH}\sqrt{2h/g} = 2\sqrt{Hh}. \tag{48}$$
   Finally we can fix a total filling height (measured from the floor) to be H_f, so that $H_f = H + h$. Where do we have to place h to maximise x? To answer this question, we maximise
$$x = 2\sqrt{(H_f - h)h}$$
   with respect to h. This gives $h = \frac{1}{2}H_f$, and so
$$x_{max} = H_f.$$

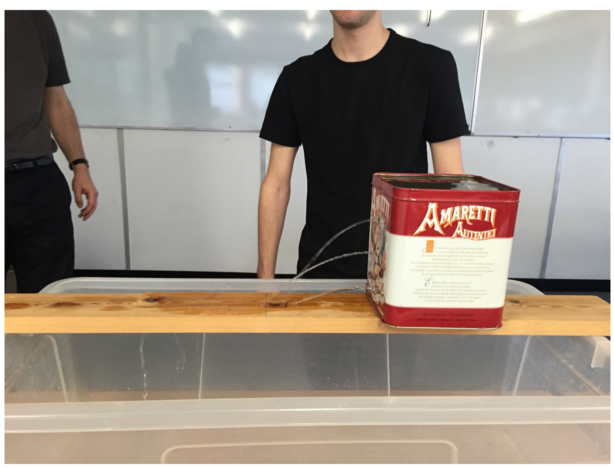


Figure 22: Experimental test of (48).

Experiment: We took data from the bottom hole, which is at $h = 2.4$cm. The length of the trajectory x was measured as for the filling height $H_f = H + h$ as given in the table:

H_f [cm]	H [cm]	$2\sqrt{(H_f - h)h}$ [cm]	x_{exp} [cm]
3.5 ± 0.1	1.1± 0.1	3.2± 0.2	2.2± 0.1
8.5 ± 0.1	6.1 ± 0.1	7.7± 0.2	6.6± 0.1
12. ± 0.1	9.6± 0.1	9.6± 0.2	8.1± 0.1

There are signifcant deviations, but they decrease for increasing H_f.

Example 2.7.3 (Flow through a slowly diverging channel) *We can use Bernoulli's equation to calculate the pressure distribution in a pipe whose cross section is changing.*

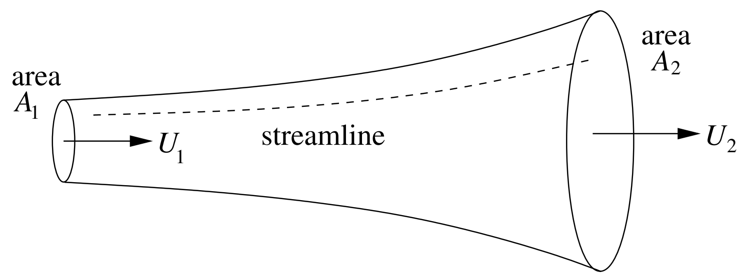


Figure 23: A slowly widening channel.

From mass conservation: $A_1U_1 = A_2U_2$, and from Bernoulli's equation (ignoring gravity):

$$\frac{1}{2}\rho U_1^2 + p_1 = const = \frac{1}{2}\rho U_2^2 + p_2.$$

Hence, the pressure drop is

$$\Delta p = p_2 - p_1 = \frac{1}{2}\rho(U_1^2 - U_2^2) = \frac{1}{2}\rho U_1^2(1 - A_1^2/A_2^2) > 0 \tag{49}$$

for a widening channel. As a channel widens, the flow slows down, and the pressure increases. In the Venturi tube (see Fig. 24), this effect is used to measure the flow rate $Q = A_1U_1$. Namely, since A_1,A_2, and Δp can be measured, we can express

$$Q = \sqrt{\frac{2\Delta p A_1^2 A_2^2}{\rho(A_2^2 - A_1^2)}}$$

in terms of measurable quantities.

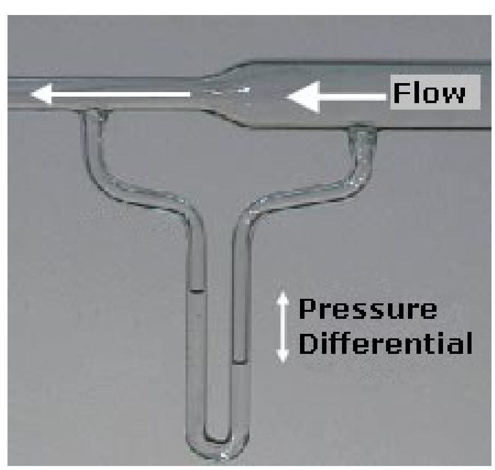


Figure 24: The Venturi tube measures the pressure difference between parts of a pipe with different cross sections.

2.8 The vorticity equation

The vorticity is the key to many phenomena in fluid mechanics. It is often concentrated into small regions of space, for example near a point (a point vortex), a line or a sheet (a vortex sheet). Understnding the motion of the vorticity thus often holds the key to the entire evolution of the flow. We start with Euler's equation having rewritten the transport term using (42), namely

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2).$$

Here we have assumed that the density ρ is constant. Taking the curl of this equation eliminates the right hand side:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = 0.$$

We can use a vector identity:

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{u}(\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \mathbf{u}) + (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega}$$

since the flow is incompressible and $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) = 0$. Hence

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u}, \tag{50}$$

or

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} \tag{51}$$

in either form, (50) or (51) are called the vorticity equation. Its advantage is that we have succeeded to eliminate the pressure from the Euler equation. However, we have payed a heavy price: (50) contains both the velocity and the vorticity fields $\mathbf{u}, \boldsymbol{\omega}$. Thus whenever $\boldsymbol{\omega}$ has been updated using (50), we need to reconstruct $\mathbf{u}$ from it.

The situation is more promising in two-dimensional flows, $\mathbf{u} = (u(x,y), v(x,y), 0)$ and so, by definition $\boldsymbol{\omega} = (0, 0, \omega(x,y)) = \omega(x,y)\hat{\mathbf{z}}$. It is then clear that the term

$$(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} = \omega(x,y)\frac{\partial \mathbf{u}}{\partial z} = 0$$

and so

$$\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{0} \tag{52}$$

That is, vorticity is conserved as it moves with the flow.

Note 2.8.1 (Conservation of vorticity in two dimensions) *If $\boldsymbol{\omega} = 0$ at time $t = 0$, then $\boldsymbol{\omega} = 0$ for all time. Vorticity cannot be generated in a 2D flow.*

2.9 Kelvin's circulation theorem

In three dimensions conservation of vorticity (which corresponds to conservation of angular momentum in mechanics) takes a somewhat more subtle form.

Definition 2.9.1 (Circulation) *The circulation of a velocity field is defined to be*

$$\Gamma = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l}, \tag{53}$$

where $C(t)$ is a closed loop which moves with the fluid.

Note 2.9.2 (relation to a patch of vorticity) *By Stokes' theorem,*

$$\Gamma = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} = \int_{S(t)} (\boldsymbol{\nabla} \times \mathbf{u}) \cdot \mathbf{n} dS = \int_{S(t)} \boldsymbol{\omega} \cdot \mathbf{n} dS,$$

where $S(t)$ is a surface whose edges connect with $C(t)$.

Theorem 2.9.3 (Kelvin's theorem) *The derivative of the circulation along a closed loop convected by the flow is constant:*

$$\frac{D\Gamma}{Dt} = 0. \tag{54}$$

Proof: Since $C(t)$ moves with the fluid, we can think of it as being made up of Lagrangian markers s, which move with the fluid. Thus $C(t)$ is defined by the curve $\mathbf{r}(s,t)$, with (say) $a \le s \le b$, and we can parameterize Γ as

$$\Gamma = \int_a^b \mathbf{u} \cdot \frac{\partial \mathbf{r}}{\partial s} ds.$$

Since s is a Lagrangian marker, $\dfrac{D\mathbf{r}(s,t)}{Dt} = \mathbf{u}$. Thus the total derivative of Γ is

$$\frac{D\Gamma}{Dt} = \int_a^b \frac{D\mathbf{u}}{Dt} \cdot \frac{\partial \mathbf{r}}{\partial s} ds + \int_a^b \mathbf{u} \cdot \frac{\partial}{\partial s} \frac{D\mathbf{r}(s,t)}{Dt} ds = -\int_a^b \boldsymbol{\nabla}(p/\rho + \Phi) \cdot \frac{\partial \mathbf{r}}{\partial s} ds + \int_a^b \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial s} ds, \tag{55}$$

where we have used Euler's equation (24) for constant ρ.

But both integrals on the right hand side of (55) vanish: The first,

$$\int_a^b \boldsymbol{\nabla}(p/\rho + \Phi) \cdot \frac{\partial \mathbf{r}}{\partial s} ds = \oint_{C(t)} \boldsymbol{\nabla}(p/\rho + \Phi) \cdot d\mathbf{l} = 0$$

is the integral of a gradient field over a closed loop. The second,

$$\int_a^b \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial s} ds = \frac{1}{2} \int_a^b \frac{\partial}{\partial s} \mathbf{u}^2 ds = 0,$$

vanishes again since s parameterizes a closed loop, and so $\mathbf{u}(a) = \mathbf{u}(b)$. Hence we obtain the result.

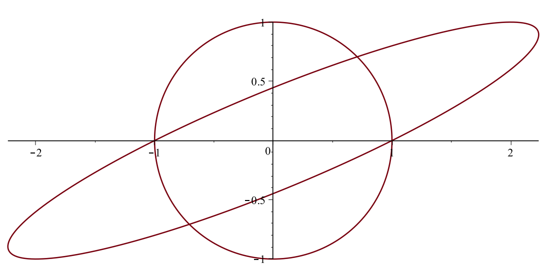


Figure 25: A circle being convected by the shear flow (56).

Example 2.9.4 (A shear flow) *Let a closed loop of particles $C(t)$ be defined by (see Fig. 25):*

$$\mathbf{r} = a(\cos s + \alpha t \sin s, \sin s, 0), \quad 0 \leq s < 2\pi,$$

where each value of s corresponds to a different fluid particle, and $a, \alpha > 0$.

On Sheet 1, Q. 1, we worked out how $C(t)$ changed in time. We also showed that the Eulerian velocity field is

$$\mathbf{u} = (\alpha y, 0, 0), \tag{56}$$

which clearly is a solution of Euler's equation.

Since s parameterizes the curve, we can calculate the circulation at any time as

$$\Gamma = \int_{C(t)} \mathbf{u} \cdot d\mathbf{l} = \int_{2\pi} \mathbf{u} \cdot \frac{\partial \mathbf{r}}{\partial s} ds.$$

Now

$$\frac{\partial \mathbf{r}}{\partial s} = a(-\sin s + \alpha t \cos s, \cos s, 0)$$

and

$$\mathbf{u}(s) = \frac{\partial \mathbf{r}}{\partial t} = a\alpha(\sin s, 0, 0),$$

so that

$$\Gamma = \alpha a^2 \int_{2\pi} \sin s(-\sin s + \alpha t \cos s) ds = -\alpha a^2 \int_{2\pi} \sin^2 s ds = -\pi \alpha a^2.$$

This is constant, consistent with Kelvin's theorem.

3 Irrotational flows: potential theory

Kelvin's circulation theorem states that the circulation around any loop convected by the flow cannot change, *if the fluid is inviscid.* In particular, if the circulation is zero, it will remain zero. Now if there is no vorticity present in the flow at some initial time, the circulation around *any* loop in the flow is zero. Hence Kelvin's circulation theorem guarantees that there will never be any circulation in the flow, and thus $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = 0$ throughout the domain. No vorticity can be created in a flow domain if the viscosity is small.

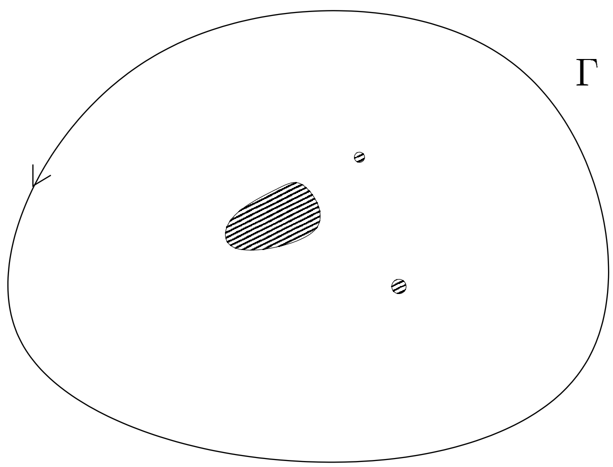


Figure 26: Patches of vorticity remain confined by a loop around them.

However, we will see in a little while that problems arise with this argument when applied near solid boundaries. The reason is that the circulation theorem applies to loops. However, it is impossible to surround a point on a solid boundary by a loop that stays in the flow. Thus no conclusions can be drawn on parts of the flow that *originate* from a solid wall. However, even in this case Kelvin's theorem has very important consequences. Imagine a localised region of vorticity (shaded region) having been created by shedding from a boundary, as seen in Fig. 26. Placing a loop Γ around it, which is convected by the flow, both are convected together, and the vorticity will stay inside the loop. Then Kelvin's theorem guarantees that the total strength remaines constant. In other words, the region of the flow which contains vorticity remains localised within a given region.

Example 3.0.5 (Circulation of a vortex) *The most important example of an isolated region of vorticity is that of a vortex, whose vorticity is confined to a point. We have already looked at this in Example 1.8.3 and on Sheet 3, Q2.*

In polar coordinates, the velocity field is $\mathbf{u} = A/r\hat{\boldsymbol{\theta}}$. Then $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = 0$ apart from at $r = 0$. But

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l} = \int_0^{2\pi} \frac{A}{r} r d\theta = 2\pi A.$$

In terms of the total circulation, the velocity field becomes

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}$$

We will concentrate on flows which either do not contain vorticity, or whose vorticity is concentrated into singular regions. In that case, very powerful tools from potential theory can be brought to bear on fluid problems, as we will see now.

3.1 The velocity potential

Theorem 3.1.1 *Existence of the velocity potential Let* $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = 0$ *throughout a flow domain* $\mathcal{D}$ *(apart from at isolated singularities). SUch a flow is called irrotational. Then there exists a scalar field* $\phi(\mathbf{r}, t)$ *such that*

$$\mathbf{u} = \boldsymbol{\nabla}\phi. \tag{57}$$

The field ϕ is called the velocity potential or simply the potential of $\mathbf{u}$. If the velocity field $\mathbf{u}$ is given, the potential is calculated from the path integral over $\mathbf{u}$,

$$\phi = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{u} \cdot d\mathbf{l},$$

provided of course the flow is indeed potential. In particular, the result is independent of the path taken, provided the domain is simply connected (see Section 4.2 below).

If the fluid is also incompressible then, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi = \boldsymbol{\nabla}^2\phi \equiv \triangle\phi = 0. \tag{58}$$

This is Laplace's equation. When the fluid domain communicates with a (moving) solid boundary, we impose the kinematic boundary condition (34), which can be written as $\mathbf{u} \cdot \mathbf{n} = f(\mathbf{r})$, or $\mathbf{n} \cdot \boldsymbol{\nabla}\phi = f(\mathbf{r})$ for some given f.

The field equation (Laplace's) and the boundary conditions are an example of a Neumann boundary-value problem. An important feature of the equations are that they are linear and so we can use superposition of solutions. This may seem odd, since the Euler equation is a nonlinear equation, and does not obey the superposition principle. However, as we will see very soon, the pressure depends on ϕ in a non-linear way (later), which reflects the nonlinear character of the Euler equation.

Theorem 3.1.2 (Uniqueness of the velocity potential) *Suppose an incompressible irrotational fluid occupies a simply connected domain* $\mathcal{D}$*, so* $\mathbf{u} = \boldsymbol{\nabla}\phi$ *and that on the boundaries* S *of* $\mathcal{D}$*,* $\mathbf{u} \cdot \mathbf{n} = f(\mathbf{r})$*. Then the potential* ϕ *is unique up to a constant.*

. **Proof:** Suppose ϕ were non-unique, so that both ϕ_1 and ϕ_2 satisfy

$$\boldsymbol{\nabla}^2\phi_i = 0, \qquad \text{and} \qquad \mathbf{n} \cdot \boldsymbol{\nabla}\phi_i = f(\mathbf{r}), \qquad \text{for } i = 1, 2$$

and $\phi_1 \neq \phi_2$. Let $\psi = \phi_1 - \phi_2$

$$\begin{aligned}\int_{\mathcal{D}} |\boldsymbol{\nabla}\psi|^2 dV \equiv \int_{\mathcal{D}} \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\psi dV &= \int_{\mathcal{D}} \boldsymbol{\nabla} \cdot (\psi\boldsymbol{\nabla}\psi) dV, \qquad \text{since } \boldsymbol{\nabla}^2\psi = 0 \\ &= \int_S \psi\boldsymbol{\nabla}\psi \cdot \mathbf{n} dS = 0, \qquad \text{using Gauss' theorem}\end{aligned}$$

since $\boldsymbol{\nabla}\psi \cdot \mathbf{n} = \boldsymbol{\nabla}\phi_1 \cdot \mathbf{n} - \boldsymbol{\nabla}\phi_1 \cdot \mathbf{n} = 0$ on S.

Since the integral over $|\boldsymbol{\nabla}\psi|^2$ vanishes, $|\boldsymbol{\nabla}\psi|^2$ must also vanish throughout the domain: if $|\boldsymbol{\nabla}\psi|^2$ were non-zero somewhere, it would make a strictly positive contribution to the integral. But sine there are only positive contributions to the integral, the total would also be strictly positive, contradicting what we showed. Hence we conclude that $|\boldsymbol{\nabla}\psi| = 0$ in $\mathcal{D}$, and so $\boldsymbol{\nabla}\psi = 0$ in $\mathcal{D}$. In a *simply connected domain* this implies that $\psi = const$ in $\mathcal{D}$. But this means that ϕ_1 and ϕ_2 only differ by a constant, which is irrelevant, as only terms in $\boldsymbol{\nabla}\phi$ appear. We will come back to the non-uniqueness that exists in multiply connected domains below.

3.2 Some simple flows and their potentials

3.2.1 Uniform flow

If the velocity field is constant, $\mathbf{u} = \mathbf{U} = const$, a simple integration yields

$$\phi = \mathbf{U} \cdot \mathbf{r}.$$

If $\mathbf{u}$ is chosen to point in the x-direction, $\mathbf{u} = U\hat{\mathbf{x}}$, then $\phi = Ux$. In two dimensions, the streamfunction would then be $\psi = Uy$.

3.2.2 A three-dimensional source

The flow field has been derived in Chapter 2, see (30). Integrating radially from infinity, we find

$$\phi = -\frac{m}{4\pi r}.$$

It is shown readily that Laplace's equation is indeed satisfied:

$$\partial_i^2 \frac{1}{r} = -\partial_i \frac{x_i}{r^3} = -\frac{3}{r^3} + 3\frac{x_i^2}{r^5} = 0.$$

Taking the gradient, one finds the velocity field:

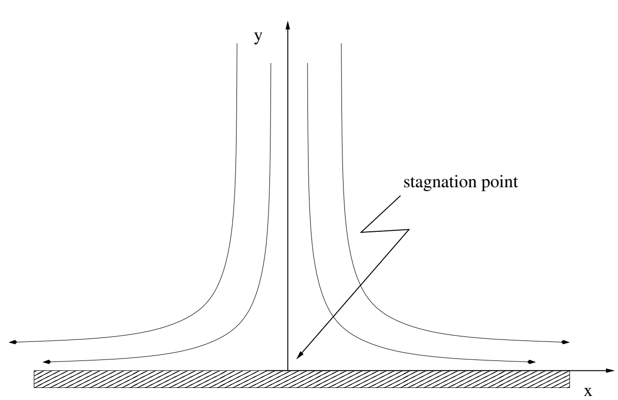
$$u_i = -\partial_i \frac{m}{4\pi r} = \frac{m x_i}{4\pi r^3},$$

and thus

$$\mathbf{u} = \frac{m\mathbf{r}}{4\pi r^3} = \frac{m}{4\pi r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the basis vector in spherical coordinates.

3.2.3 Stagnation point flow



We begin with the two-dimensional case of a flow onto a flat plate, see Figure. We consider the simplest case in which the flow is two-dimensional, $\mathbf{u} = (u, v, 0)$. There must be a stagnation point $\mathbf{u} = 0$ somewhere along the plate, which we place at the origin, $x = y = 0$. Performing a Taylor expansion about the stagnation point, to lowest order the x-component of the velocity will increase linearly as one moves away from the line of symmetry: $u = Ax$. On account of incompressibility, we have

$$A = \frac{\partial Ax}{\partial x} = -\frac{\partial v}{\partial y},$$

and thus $v = -Ay$ if one moves away from the plate against the flow. Integrating, one finds the potential:

$$\phi = \frac{A}{2}(x^2 - y^2) \tag{59}$$

It is easy to confirm that this is a solution of Laplace's equation, and thus satisfies the fluid equations. More importantly, since (59) was derived from a local expansion, it is valid locally near *any* two-dimensional stagnation point, regardless of the shape of the surface, or of the exterior flow. The stream function corresponding to (59) is

$$\psi = Axy, \tag{60}$$

as is confirmed directly from (17).

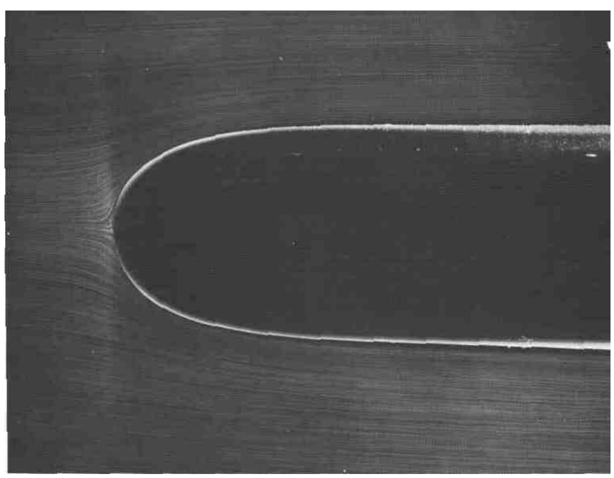


Figure 27: An axisymmetric flow around a so-called Rankine body, showing a stagnation point near the tip of thre body to the left. The flow in the neighborhood of this point will be described by (61).

The axisymmetric version of stagnation point flow is relevant if any axisymmetric body is placed in an oncoming stream. The potential is in that case (exercise)

$$\phi = \frac{A}{4}(r^2 - 2z^2). \tag{61}$$

3.3 Bernoulli's theorem for unsteady, irrotational flows

Within potential flow theory, we have the slightly confusing situation that we are seemingly solving flow problems only by solving Laplace's equation, which in turn comes from the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. We are *not* using Euler's equation of motion (25) to solve flow problems (at least this is almost true; we have been using (25) indirectly inasmuch we used it to derive Kelvin's theorem, which motivated the potential flow assumption).

However, the Euler equation *does* come in, albeit in a sneaky way, as we will now show. According to (43), the Euler equation can be written

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2).$$

For irrotational flows, $\mathbf{u} = \nabla\phi$ and $\boldsymbol{\omega} = 0$, and so

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \Phi + \tfrac{1}{2}\mathbf{u}^2\right) = 0.$$

It follows that

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \Phi + \tfrac{1}{2}\mathbf{u}^2 = C(t), \tag{62}$$

where $C(t)$ is an arbitrary function of time.

Remark 3.3.1 (Unsteady Bernoulli equation)

1. *Note that (62) applies throughout the fluid, not just on a streamline.*
2. *By defining a new velocity potential*

$$\tilde{\phi} = \phi - \int^t C(t')dt',$$

the function $C(t)$ can always be eliminated from the problem. The potential $\tilde{\phi}$ leads to the same flow field! (why?)

The Bernoulli equation (62) is best viewed as an equation for the pressure. Thus solving a potential flow problem then becomes a two-step process:

1. Solve Laplance's equation for ϕ, using the boundary conditions $\mathbf{n} \cdot \nabla\phi = f(\mathbf{r})$ on the boundary of $\mathcal{D}$.
2. Given ϕ, compute $\mathbf{u} = \nabla\phi$ and then (62) to compute the pressure. The pressure depends on $\mathbf{u}$ in a non-linear way, a reflection of the nonlinearity of the Euler equation.

3.4 The collapse of a spherical cavity

As a first application of this program, consider a spherical cavity that has opened up in a large expanse of liquid. Such holes or "cavitation bubbles" are produced quite commonly near ships' propellors, where the force of the propellor "rips apart" water molecules and leaves a small hole behind. Thus there is little pressure inside the cavity, but a large pressure in the surrounding fluid, which forces the hole to close again. Since the hole is small, it is justified to neglect gravity, and to consider a constant positive pressure $p_0 > 0$ far from the cavity, while the pressure inside the cavity vanishes. For simplicity, we suppose that at $t = 0$ we have a spherical bubble, initially at rest, with radius R_0 in an infinite fluid. The pressure difference causes the bubble to collapse, so $R(t) < R_0$, where $R(t)$ is radius at time t. We want to find $R(t)$ and the time of collapse t_c.

As the bubble collapses, to the outside fluid it acts like a sink at its centre. Thus we try the corresponding potential

$$\phi(r,t) = \frac{A(t)}{r},$$

where $A(t)$ is to be determined. We have seen that $\triangle\phi = 0$. Now

$$u_r = \frac{\partial \phi}{\partial r} = -\frac{A}{r^2},$$

and the boundary condition is

$$u_r = \dot{R} \equiv \frac{\mathrm{d}R}{\mathrm{d}t}, \qquad \text{on } r = R, \qquad \Rightarrow \qquad A = -R^2\dot{R},$$

and hence

$$\phi(r,t) = -\frac{R^2\dot{R}}{r}. \tag{63}$$

To find $R(t)$, we use the unsteady Bernoulli equation, so that

$$\rho\frac{\partial \phi}{\partial t} + p + \tfrac{1}{2}\rho\mathbf{u}^2 = C(t) = p_0,$$

since $\phi \to 0$ and $\mathbf{u} \to 0$ as $r \to \infty$. Now, on surface of bubble $p = 0$ (by assumption) and

$$|\mathbf{u}|^2 = \dot{R}^2, \qquad \text{and} \qquad \frac{\partial \phi}{\partial t} = -\frac{R^2\ddot{R} + 2R\dot{R}^2}{R}.$$

Substituting into Bernoulli gives

$$-R\ddot{R} - 2\dot{R}^2 + \dot{R}^2/2 = -R\ddot{R} - 3\dot{R}^2/2 = p_0/\rho,$$

which is an ODE for $R(t)$. To integrate it, we multiply by $2R^2\dot{R}$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(R^3\dot{R}^2\right) = 3R^2\dot{R}^3 + 2\ddot{R}\dot{R}R^3 = -\frac{p_0}{\rho}2R^2\dot{R} = -\frac{2p_0}{3\rho}\frac{\mathrm{d}R^3}{\mathrm{d}t}.$$

Integrating, we find

$$R^3\dot{R}^2 = -\frac{2}{3}\frac{p_0}{\rho}R^3 + C,$$

where $C = 2R_0^3 p_0/(3\rho)$ is a constant, determined by the initial condition $\dot{R} = 0$ when $R = R_0$. Hence

$$\dot{R}^2 = \frac{2}{3}\frac{p_0}{\rho}\left(\frac{R_0^3}{R^3} - 1\right) \tag{64}$$

or

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\sqrt{\frac{2}{3}\frac{p_0}{\rho}}\left(\frac{R_0^3 - R^3}{R^3}\right)^{1/2},$$

where the negative sign must be chosen because $\dot{R} < 0$ by assumption. Integrating up gives

$$\int_{R_0}^{R(t)} \frac{R^{3/2}\,\mathrm{d}R}{(R_0^3 - R^3)^{1/2}} = -\int_0^t \left(\frac{2}{3}\frac{p_0}{\rho}\right)^{1/2} \mathrm{d}t = -t\left(\frac{2}{3}\frac{p_0}{\rho}\right)^{1/2}.$$

If the bubble collapses at time $t = t_c$, then $R(t_c) = 0$ implies ($R = R_0 u$)

$$\int_0^1 \frac{u^{3/2}\,\mathrm{d}u}{(1 - u^3)^{1/2}} = \frac{t_c}{R_0}\left(\frac{2}{3}\frac{p_0}{\rho}\right)^{1/2} \tag{65}$$

The integral on the left-had side of (65) can be evaluated to give

$$\int_0^1 \frac{u^{3/2}\,\mathrm{d}u}{(1 - u^3)^{1/2}} = \frac{\sqrt{3}}{2\sqrt{\pi}}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{2}{3}\right) \approx 0.747.$$

Hence the collapse time is

$$t_c = 0.915\left(\frac{\rho}{p_0}\right)^{1/2} R_0. \tag{66}$$

We can also analyse the asymptotic behaviour of the collapse during its final moments, when the radius goes to zero. Such singular events are usually described by power laws, so we try

$$R = B(t_c - t)^{\alpha}.$$

Now $\dot{R}^2 \propto (t_c - t)^{2\alpha - 2}$, and the right hand side of (64) scales as $R^{-3} \propto (t_c - t)^{-3\alpha}$. Thus $2\alpha - 2 = -3\alpha$, and the power law exponent is found to be

$$\alpha = \frac{2}{5}.$$

Plugging this back into (64), we find

$$R = \left(\frac{2p_0 R_0^3}{3\rho}\right)^{1/5} (t_c - t)^{2/5},$$

in the limit of R approaching zero.

What is remarkable about the collapse is that the inertia of the surrounding liquid is focused into a point by the converging motion. If one calculates the *speed* of the collapse, one finds

$$\dot{R} \propto (t_c - t)^{-3/5},$$

which is *diverging* as the radius goes to zero.

If the cavity is filled with gas, the gas can be compressed to very high pressures during the last moments, reaching around 10,000 K in the interior of the bubble. This may cause ionisation of the gas, which may start to glow. This phenomenon is known as sonoluminescence, since the collapse of the bubble is started off by an external sound field.

3.5 Flow past a sphere

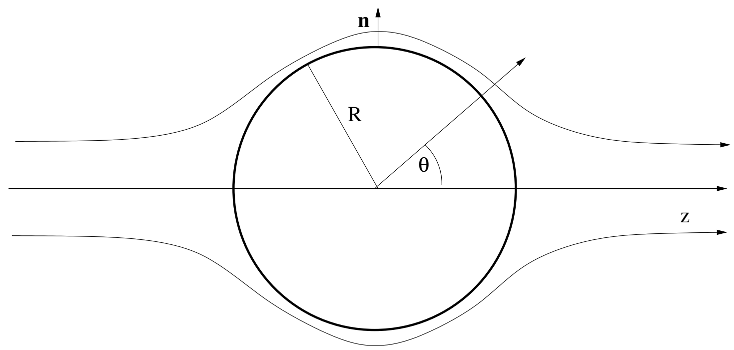


Figure 28: Schematic of the flow around a sphere in a steady stream.

One of the most fundamental problems of fluid mechanics is to understand the flow around obstacles, which stand in the way of a flow. Equivalently (think of Galilean invariance) one can think of a body moving inside a fluid which is at rest (for example an aeroplane). Let us begin studying this problem by considering the perhaps simplest possible body, a sphere of radius R. The sphere is placed inside a uniform stream, which we choose in the direction of the z or symmetry axis: $\mathbf{U} = U\hat{\mathbf{z}}$.

Far from the body, the flow will be uniform. The sphere introduces a perturbation to the flow which decays as $r \to \infty$, i.e. as one moves away from the body. According to the *superposition principle* (the Laplace equation is a linear equation), this perturbation is to be added to the uniform flow:

$$\mathbf{u} = \mathbf{U} + \boldsymbol{\nabla}\phi, \tag{67}$$

where ϕ is the potential of the perturbation. The simplest such flow we know is a source, with potential $\phi \propto 1/r$. However, such a flow cannot describe the physical situation at hand. Far from the sphere, the total mass flow through a closed control surface should be zero, not finite as for a source. Another way of putting it is that the flow field of a source decays too slowly (like $1/r^2$) away from the body.

This can be helped by taking the derivative of the source solution. Taking the derivative in the $\boldsymbol{\mu}$-direction, one arrives at the potential

$$\phi = \boldsymbol{\mu} \cdot \boldsymbol{\nabla}\frac{1}{r} = -\frac{\boldsymbol{\mu} \cdot \mathbf{r}}{r^3}, \tag{68}$$

which is called the dipole flow (why?). The dipole is oriented in the $\boldsymbol{\mu}$-direction and has strength $|\boldsymbol{\mu}|$. This flow is also a solution of Laplace's equation, since

$$\triangle\left(\boldsymbol{\mu} \cdot \boldsymbol{\nabla}\frac{1}{r}\right) = \boldsymbol{\mu} \cdot \boldsymbol{\nabla}\triangle\left(\frac{1}{r}\right) = 0.$$

In summary, from (67) and (68) our ansatz for the velocity is

$$u_i = U\delta_{i3} + \partial_i\phi = U\delta_{i3} - \partial_i\frac{\mu_j x_j}{r^3} = U\delta_{i3} - \frac{\mu_i}{r^3} + 3\mu_j x_j\frac{x_i}{r^5},$$

or

$$\mathbf{u} = U\hat{\mathbf{z}} - \frac{1}{r^3}\left(\boldsymbol{\mu} - 3\boldsymbol{\mu} \cdot \mathbf{r}\frac{\mathbf{r}}{r^2}\right). \tag{69}$$

Now we have to satisfy the boundary condition on the surface of the sphere, which is $\mathbf{u}\cdot\mathbf{n} = 0$. The normal vector to the sphere is $\mathbf{n} = \mathbf{r}/R$. Now on the surface of the sphere $(r = R)$:

$$\mathbf{u}\cdot\mathbf{n} = U(\hat{\mathbf{z}}\cdot\mathbf{n}) - \frac{1}{R^3}\left(\boldsymbol{\mu}\cdot\mathbf{n} - 3\frac{\boldsymbol{\mu}\cdot\mathbf{n}}{R^3}\right) = U(\hat{\mathbf{z}}\cdot\mathbf{n}) + 2\frac{\boldsymbol{\mu}\cdot\mathbf{n}}{R^3},$$

so choice $\boldsymbol{\mu} = -\frac{UR^3}{2}\hat{\mathbf{z}}$ will guarantee that this is zero for $r = R$. In summary, the velocity field which satisfies all the boundary conditions is $(z = \hat{\mathbf{z}}\cdot\mathbf{r})$:

$$\mathbf{u} = U\hat{\mathbf{z}} + \frac{U}{2}\left(\frac{R}{r}\right)^3\left(\hat{\mathbf{z}} - 3\frac{z}{r}\mathbf{n}\right). \tag{70}$$

Now we want to calculate the pressure field, and in particular the pressure on the surface of the sphere. This will permit us to calculate the total force on the sphere. On the surface we have

$$\mathbf{u} = \frac{3U}{2}\left(\hat{\mathbf{z}} - \frac{z}{R}\mathbf{n}\right),$$

so

$$\mathbf{u}^2 = \frac{9U^2}{4}\left(1 - 2\frac{z}{R}\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} + \frac{z^2}{R^2}\right) = \frac{9U^2}{4}\left(1 - \frac{z^2}{R^2}\right) = \frac{9U^2}{4}\left(1 - \cos^2\theta\right),$$

using that $\frac{z}{r} = \cos\theta$.

According to Bernoulli's equation

$$p + \rho\frac{\mathbf{u}^2}{2} = C(t),$$

everywhere in the fluid. But the pressure is $p = p_{atm}$ far from the sphere, and the flow is uniform: $\mathbf{u}^2 = U^2$, which implies that $C = p_{atm} + \rho U^2/2$. As a result, the pressure on the surface of the sphere is

$$p = p_{atm} + \frac{\rho(U^2 - \mathbf{u}^2)}{2} = p_{atm} + \frac{\rho U^2}{8}\left(9\cos^2\theta - 5\right), \tag{71}$$

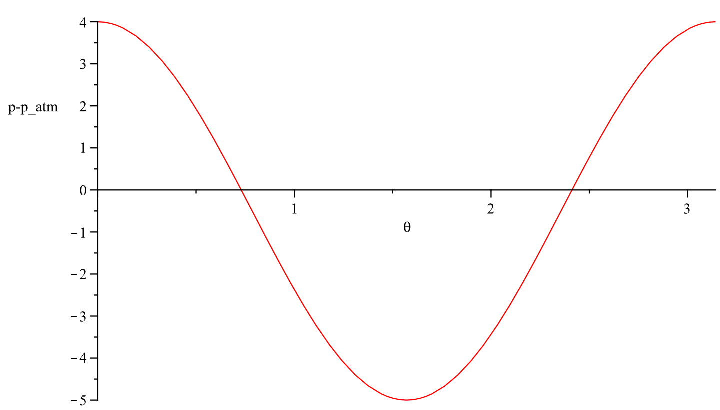


Figure 29: The pressure distribution around a sphere as function of the angle.

This result has the remarkable property that the pressure distribution is *symmetric* around the equator of the sphere $\theta = \pi/2$. Thus while the sphere is pushed by the flow in the downstream direction, there is an equally high overpressure in the back. As a result, without further calculation, we can conclude that the total *drag force* on the sphere is zero. By definition, the drag on a body in a uniform stream is the force in the direction of the stream. The force perpendicular to the stream is called the lift, which is of course zero if the flow is axisymmetric, as is the case here.

Let us check these statements by doing the calcuation explicitly. Integrating over the surface of the sphere, the total force is

$$\mathbf{F} = -\int_{r=R} p\mathbf{n}dS = -R^2 \int_0^{2\pi} \int_0^{\pi} p\mathbf{n} \sin\theta d\theta d\phi,$$

where $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the normal in spherical coordinates. Performing the ϕ-integral over the x and y components, we obtain

$$\int_0^{2\pi} \cos\phi d\phi = \int_0^{2\pi} \sin\phi d\phi = 0,$$

so the first two components of $\mathbf{F}$ vanish. This is clear because on account of the symmetry of the problem, there can be no component normal to the direction of the flow, which is in the z-direction. It remains to calculate the force in the flow direction ($\mathbf{n} \cdot \hat{\mathbf{z}} = \cos\theta$):

$$F_z = \mathbf{F} \cdot \mathbf{z} = -R^2 \int_0^{2\pi} \int_0^{\pi} p \cos\theta \sin\theta d\theta d\phi = -2\pi R^2 \frac{\rho U^2}{8} \int_0^{\pi} \left(9\cos^2\theta - 5\right) \cos\theta \sin\theta d\theta =$$

$$-\frac{\pi}{4}\rho R^2 U^2 \int_{-1}^{1} \left(9x^2 - 5\right) x dx = -\frac{\pi}{4}\rho R^2 U^2 \left[\frac{9}{4}x^4 - \frac{5}{2}x^2\right]_{-1}^{1} = 0,$$

as claimed. In conclusion, $\mathbf{F} = 0$, all components of the force vanish!

3.6 d'Alambert's paradox

At first sight, one might think that the fact that there is no force acting on a sphere in a stream $\mathbf{U}$ *at steady state* is related to the high symmetry of the sphere. In fact, this is not the case: we now show that the force $\mathbf{F}$ acting on a body of *arbitrary* shape vanishes at steady state. At least as far as the component of $\mathbf{F}$ in the direction of $\mathbf{U}$ is concerned, this is a statement about energy conservation, since $\mathbf{F} \cdot \mathbf{U}$ is the power. Since there is no friction, this energy input cannot be dissipated; the only other possible option would be for the energy to be transported off to infinity. The fact that this cannot occur is a statement about how fast the perturbation introduced by the body into the stream decays at infinity.

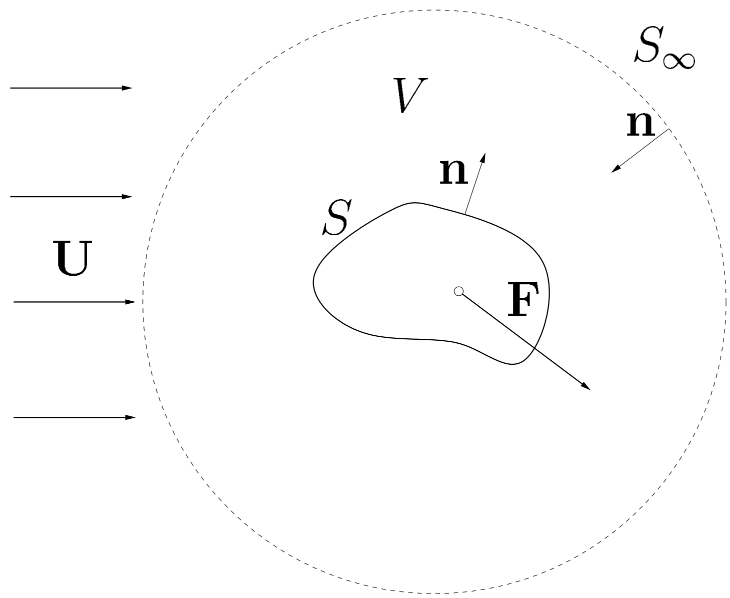


Figure 30: The force on an arbitrary body in a uniform stream.

Let us write the total velocity field in the form

$$\mathbf{u} = \mathbf{U} + \mathbf{v},$$

where $\mathbf{v} = \boldsymbol{\nabla}\phi$ is the perturbation introduced by the body. As argued before for the case of the sphere, we will assume that

$$\phi = O\left(r^{-2}\right) \text{ for } r \to \infty,$$

since the solution that decays like $1/r$ is a source or a sink. Now the force on the body is calculated from integrating the pressure force over the body:

$$\mathbf{F} = -\int_S p\mathbf{n}dS;$$

the minus sign comes from the normal pointing outward, opposite the direction of the pressure force. To find p, we use the Bernoulli equation (62) with $\dfrac{\partial\phi}{\partial t} = 0$, since the flow is steady:

$$p + \rho U^2/2 + \rho\mathbf{v}^2/2 + \rho\mathbf{U}\cdot\mathbf{v} = p_{atm} + \rho U^2/2.$$

In other words,

$$\mathbf{F} = -\int_S p_{atm}\mathbf{n}dS + \frac{\rho}{2}\int_S \mathbf{v}^2\mathbf{n}dS + \rho\int_S (\mathbf{U}\cdot\mathbf{v})\mathbf{n}dS.$$

The first integral on the right is zero, as we have seen before. As a mathematical corollary, we will show below that

$$\int_S \mathbf{v}^2\mathbf{n}dS = 2\int_S \mathbf{v}(\mathbf{v}\cdot\mathbf{n})dS. \tag{72}$$

But on the surface of the body the normal component of the total velocity vanishes: $\mathbf{n}\cdot(\mathbf{U}+\mathbf{v}) = 0$, so the force becomes

$$\mathbf{F} = \rho\int_S \left[\mathbf{v}(\mathbf{v}\cdot\mathbf{n}) + (\mathbf{U}\cdot\mathbf{v})\mathbf{n}\right]dS = \mathbf{F} = -\rho\int_S \left[\mathbf{v}(\mathbf{U}\cdot\mathbf{n}) - (\mathbf{U}\cdot\mathbf{v})\mathbf{n}\right]dS.$$

In components, this means that

$$F_i = -\rho U_j \int_S (v_i n_j - v_j n_i)\, dS. \tag{73}$$

From the symmetry of this expression it is clear immediately that the *drag* of the body, which is defined as

$$D = \mathbf{F} \cdot \mathbf{U}$$

is zero. However, the components of $\mathbf{F}$ normal to $\mathbf{U}$ are also zero. Let V be the volume exterior to the body but bounded by a closed surface S_∞ very far from the body (see Figure). Then it follows from the divergence theorem that

$$\int_V \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right) dV = -\int_S (v_i n_j - v_j n_i)\, dS - \int_{S_\infty} (v_i n_j - v_j n_i)\, dS;$$

note that $\mathbf{n}$ points *into* the volume V. If $\mathbf{v}$ decays faster than $1/r^2$ at infinity, the integral over S_∞ goes to zero as the radius of S_∞ goes to infinity. As for the integrand of the volume integral on the left, it clearly vanishes if $i = j$. If $i \neq j$, then the integrand is (up to a sign) ω_k where $k \neq i, j$. But since the flow is potential, this is again zero. It follows that the surface integral appearing on the left hand side of (73) is zero, and thus the force is altogether zero.

The corollary

We now demonstrate (72). From the divergence theorem we conclude that

$$\int_S \mathbf{v}^2 n_i dS = -\int_V \partial_i \mathbf{v}^2 dV - \int_{S_\infty} \mathbf{v}^2 n_i dS$$

The integrand of the second integral on the right behaves like $1/r^6$, so its contribution vanishes. To deal with the volume integral, we note that according to Appendix A, $\boldsymbol{\nabla}\mathbf{v}^2 = 2(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + 2\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})$, and so $\boldsymbol{\nabla}\mathbf{v}^2 = 2(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ for an irrotational flow. Since the fluid is incompressible, we also have $((\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v})_i = v_j \partial_j v_i = \partial_j (v_i v_j)$. Thus

$$\partial_i \mathbf{v}^2 = 2\partial_j (v_i v_j),$$

and we can write

$$\int_S \mathbf{v}^2 n_i dS = -\int_V \partial_i \mathbf{v}^2 dV = -2\int_V \partial_j (v_i v_j)\, dV = 2\int_S v_i v_j n_j dS,$$

using the divergence theorem in the last step. But this proves the corollary.

This is a truly remarkable result of potential flow theory, but it also presents a series problem, as absence of drag is clearly not in accord with observation. If our argument were completely airtight, airline companies would surely be doing some serious overcharging! (maybe they do anyway).

3.7 Drag

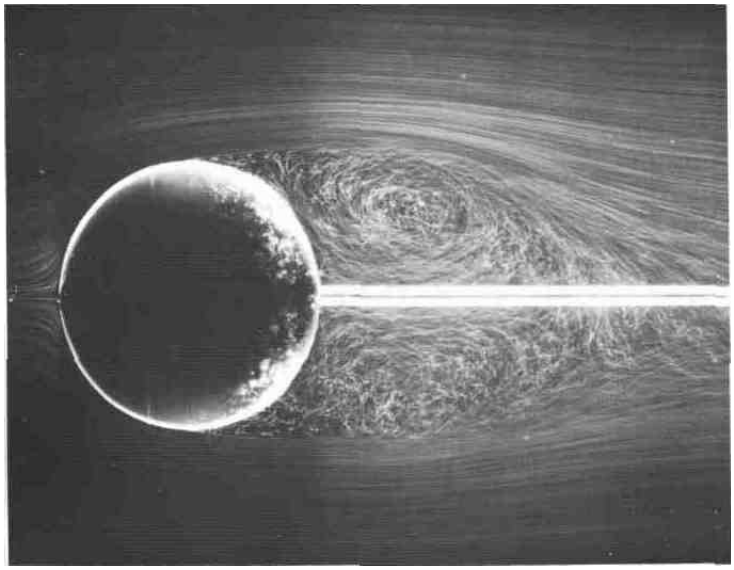


Figure 31: A wake of almost stagnant fluid forms behind an obstacle in a strong flow.

The experimental picture of a sphere in a stream reveals what the problem is: the flow around the sphere is *far* from rear-aft symmetric. While the flow in front of the sphere is nicely attached to the body and well described by (70), the back is very different. The so-called *Reynolds number* of the flow is

$$Re = \frac{UR\rho}{\eta} \approx 1.5 \times 10^4 \tag{74}$$

which is a dimensionless measure of the size of the viscosity η. Since the Reynolds number is very large, one might believe that viscous effects are small and the potential flow assumption represents a faithful representation. However, this is not the case. In particular, the flow does not remain attached to the body, and rather there are fluid particles which originate from the surface of the sphere and the enter the liquid. This is also the place where *vorticity* enters the flow. Clearly, vortices have formed in so-called *wake* of the flow.

Apart from the vortices, the wake appears to be a relatively stagnant region of the flow, which is thus at constant pressure $p = p_{cav}$, provided it remains closed (which it seems to be in the experimental Figure). To estimate the value of p_{cav}, we assume that the pressure is continuous across the point of separation.

It is also interesting to consider the streamline that separates from the body and bounds the cavity. Since the pressure is constant in the cavity, the pressure on this streamline is also constant, by continuity of the pressure. Such a streamline is called a *free streamline.* According to the steady Bernoulli equation (44) the velocity along this streamline is also constant. Therefore, there is a **jump** of the velocity at the free streamline to the almost vanishing velocity in the wake. (Nothing prevents the velocity to be discontinuous if there is no viscosity). Such a surface over which the velocity changes abruptly is called a shear layer or a *vortex sheet.* Now pressure in the cavity will be the pressure on the separating streamline, as it leaves the body.

To do any calculation of the effect of the wake, one needs an estimate of the point where the flow leaves the body, to produce the wake. A very simple criterion (which can be justified in more detail using full viscous theory), is the concept of the "adverse pressure gradient". At the front of the body the pressure maximum, and then decreases to a minimum value at the equator, where the speed is maximum; from there the pressure decreases again. It is intuitive and can be motivated in greater detail from the viscous flow theory, that the flow along the body is stable as long as the pressure decreases along the path of a fluid particle. If however a fluid article has to push *against* an increasing pressure ("adverse pressure gradient"), it prefers to leave the surface of the body.

Thus on the basis of the pressure distribution which we have calculated, one can give an estimate of the place where separation is going to occur, which is at $\theta = \pi/2$ in the case of a sphere. Now we can attempt to recalculate the drag force on the sphere, assuming separation at $\theta_s = \pi/2$. The pressure in the cavity will be

$$p_{cav} = p_{atm} - \frac{5}{8}\rho U^2 :$$

in our calculation, there is an underpressure in the cavity which pulls the sphere along.

We are now in a position to calculate the drag on the sphere, doing the front and the back separately. The result cannot depend on p_{atm} which exerts equal forces on front and back; therefore, we can put it to zero. Since $\mathbf{n} \cdot \hat{\mathbf{z}} = cos\theta$, the force from the front in the flow direction is

$$F_z^{(front)} = -\int_{front} p(\theta)\cos\theta dS = -\frac{1}{8}\rho U^2 R^2 \int_0^{2\pi}\int_{\pi/2}^{\pi} \sin\theta\cos\theta(9\cos^2\theta - 5)d\theta d\phi =$$
$$\frac{\pi}{4}\rho U^2 R^2 \int_0^1 x(9x^2 - 5)dx = -\frac{\pi}{16}\rho U^2 R^2,$$

and from the back

$$F_z^{(back)} = -R^2 \int_0^{2\pi}\int_0^{\pi/2} \sin\theta\cos\theta p_{cav} d\theta = \frac{5\pi}{4}\rho U^2 R^2 \int_0^1 x dx = \frac{5\pi}{8}\rho U^2 R^2.$$

Thus the total force or drag is

$$D = F_z = \frac{9\pi}{16}\rho U^2 R^2 = \frac{9}{16}\rho A U^2,$$

where $A = \pi R^2$ is the projected area of the sphere.

For a general body, the drag coefficient C_d is defined by

$$D = \frac{\rho}{2} U^2 A C_d,$$

and so $C_d = 9/8$ for our calculation of the sphere. We hasten to add that our calculation is not particularly good, see the Figure. In the relevant range of Reynolds numbers, $C_D \approx 0.5$. Our intention is to give a general idea of where the drag is coming from, and how d'Alambert's paradox arises.

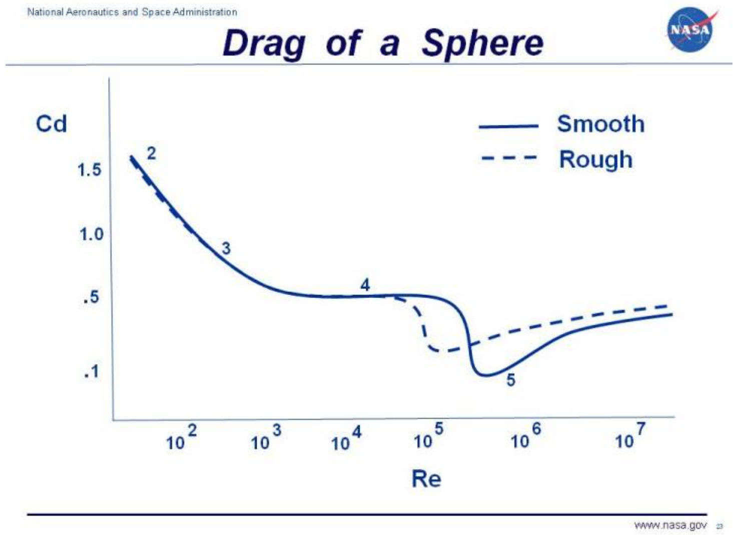


Figure 32: The drag coefficient C_D of a sphere as a function of Reynolds number.

4 Two-dimensional flows

If the flow is in the plane, and there are only two independent variables x, y, the flow problem is much simplified. Moreover, we will see that some of the physics is fundamentally different from three dimensions. One fact we know of already is that no vorticity is created in two dimensions, $\boldsymbol{\omega}$ is simply convected with the flow. Of course, real flows are never truly two-dimensional; however if the geometry extends very far in one direction (think of the wing of an aeroplane), a two-dimensional description is a good approximation.

4.1 Flow past a cylinder

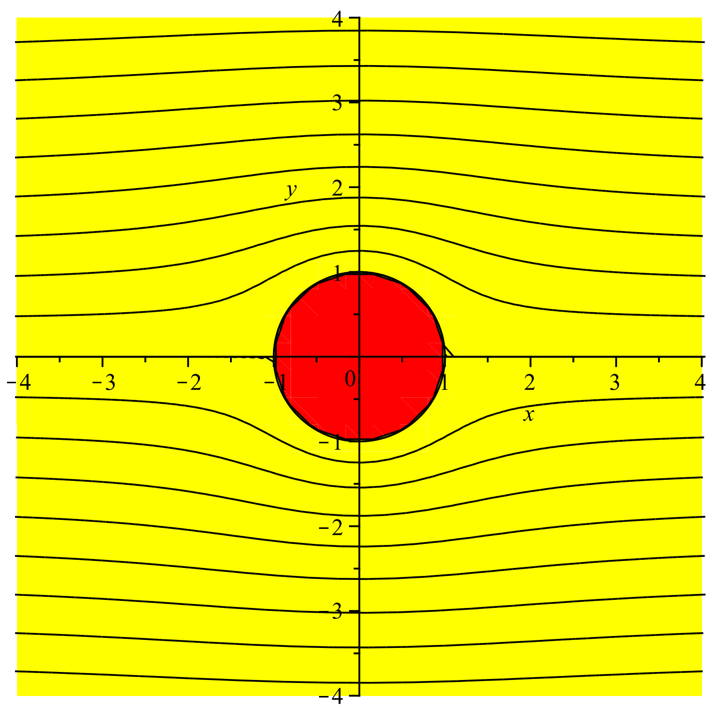


Figure 33: The streamlines around a cylinder in a uniform flow.

We want to find the flow around a stationary cylinder in a steady stream $\mathbf{U} = U\hat{\mathbf{x}}$. This is very closely analogous to the flow around a sphere; the effect of the sphere is modelled by a (two-dimensional) dipole, which is the derivative of a source, whose velocity field is given by (21). Thus the corresponding potential is

$$\phi = \frac{m}{2\pi} \ln r, \tag{75}$$

and a dipole in the $\boldsymbol{\mu}$-direction has potential

$$\phi = (\boldsymbol{\mu} \cdot \boldsymbol{\nabla}) \ln r = \frac{\boldsymbol{\mu} \cdot \mathbf{r}}{r^2}.$$

The ansatz for the velocity field is thus

$$\mathbf{u} = U\hat{\mathbf{x}} + \boldsymbol{\nabla}\phi = U\hat{\mathbf{x}} + \frac{\boldsymbol{\mu}}{r^2} - \frac{2(\boldsymbol{\mu} \cdot \mathbf{r})\mathbf{r}}{r^4}.$$

Now the boundary condition for $r = R$ is $\mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{r}/R = 0$, and from the velocity field we find ($r = R$):

$$\mathbf{u} \cdot \mathbf{n} = U\frac{x}{R} - \frac{\boldsymbol{\mu} \cdot \boldsymbol{x}}{R^3}.$$

Thus the boundary condition is satisfied if we choose

$$\boldsymbol{\mu} = R^2 U \hat{\mathbf{x}} = R^2 \mathbf{U}.$$

Thus the final answer for the velocity field is

$$\mathbf{u} = \mathbf{U} + \frac{R^2}{r^2}[\mathbf{U} - 2\mathbf{n}(\mathbf{U} \cdot \mathbf{n})]. \tag{76}$$

Now we plug this into Bernoulli's equation to compute the pressure:

$$\frac{u^2}{2} + \frac{p}{\rho} = \frac{U^2}{2} + \frac{p_{atm}}{\rho}.$$

On the surface

$$\boldsymbol{u}^2\Big|_{r=R} = (2\boldsymbol{U} - 2\boldsymbol{n}(\boldsymbol{U} \cdot \boldsymbol{n}))^2 = 4\boldsymbol{U}^2 - 4(\boldsymbol{U} \cdot \boldsymbol{n})^2 = 4U^2(1 - \cos^2\theta) = 4U^2 \sin^2\theta,$$

so in other words

$$p = p_{atm} + \frac{\rho U^2}{2}(1 - 4\sin^2\theta). \tag{77}$$

The pressure is once more *symmetric* about $\theta = \pi/2$, i.e. about the midsection of the cylinder. It follows that the total force on the cylinder is again zero.

4.2 Non-uniqueness of the potential

One particularly important aspect of two-dimensional flow is that any solid body placed in the flow domain will create a domain that is no longer simply connected.

Definition 4.2.1 (Simply connected domain) *A closed curve C is reducible in a domain $\mathcal{D}$ if it can be shrunk to a point without ever leaving $\mathcal{D}$. If every closed curve is reducible then $\mathcal{D}$ is simply connected.*

Example 4.2.2 (Simply connected domains)

1. *If $\mathcal{D}$ is the interior of a circle, then it is simply connected.*

2. *If $\mathcal{D}$ is the exterior of a circle then it is not simply connected.*

3. *Let $\mathcal{D}$ be the domain $a < r < b, 0 < \theta < 2\pi$ in cylindrical polars. $\mathcal{D}$ is not simply connected.*

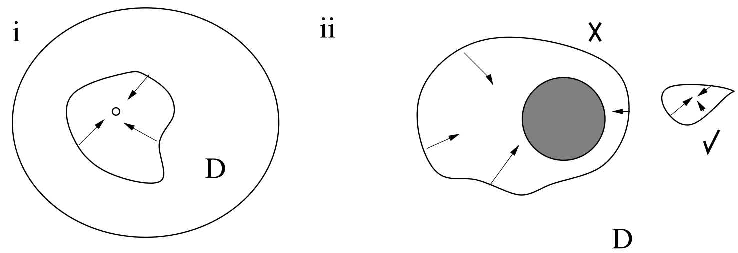


Figure 34: On the left, and example of a simply connected domain: all curves in $\mathcal{D}$ can be collapsed to a point. On the right, a domain which is not simply connected: some curves can be collapsed to a point, but those going around the hole cannot.

We now place a point **vortex** (introduced at the beginning of chapter 3) at the centre, which creates a flow

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}. \tag{78}$$

The strength of the vortex is measured in terms of its circulation Γ. Since the flow is irrotational, there exists a velocity potential s.t.

$$u_r = \frac{\partial \phi}{\partial r} = 0, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \Gamma/(2\pi r).$$

The solution is

$$\phi(r,\theta) = \frac{\Gamma}{2\pi}(\theta + A), \tag{79}$$

for any constant A.

The boundary conditions at the outer and inner walls of $\mathcal{D}$ are $\mathbf{u}\cdot\mathbf{n} = \mathbf{u}\cdot\hat{\mathbf{r}} = u_r = 0$. Clearly, (78) satisfies the boundary conditions, while the circulation Γ can assume any value: the solution is not unique! This is related to another problem with the potential: for example, $\phi(r,0) \neq \phi(r,2\pi)$ so the potential is discontinuous along $\theta = 0$ or, alternatively, ϕ is multivalued. Evidently, whenever there is circulation

$$\Gamma = \oint_C \boldsymbol{\nabla}\phi \cdot d\mathbf{l} \neq 0,$$

a finite amount of potential is picked up as one goes around the curve. The potential of such a flow is thus necessarily multivalued. This situation can arise only in a multiply connected domain: if it were simply connected, the closed curve C could be reduced to a point, and thus the integral would be zero. This can be achieved by placing a branch-cut in $\mathcal{D}$ to make the domain simply connected. In simply-connected domains, ϕ is unique, as seen in Fig. 35.

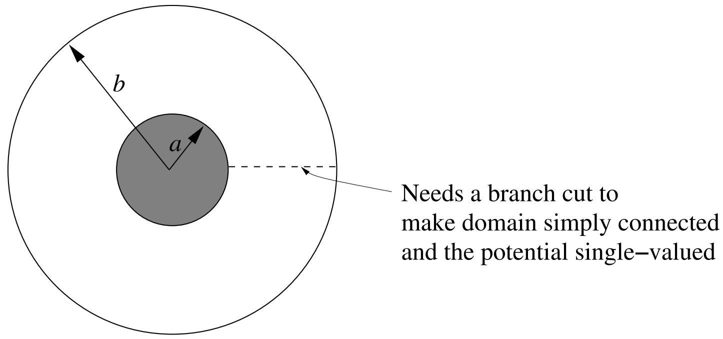


Figure 35: A branch cut avoids the non-uniqueness of the potential

Now assume *any* flow field bewteen the two cylinders, which must satisfy $\triangle\phi$ and $\mathbf{n}\cdot\boldsymbol{\nabla}\phi = 0$ on the boundaries. To any such solution, we can add the vortex (79), for any value of the circulation Γ. The result will be another solution, which also satisfies the boundary conditions. In other words, the flow between the cylinders can have any value of the circulation. In general, we see that the flow in a non-simply connected domain is determined only up to a flow with circulation Γ, where Γ can take any value.

4.3 Steady flow past a circular cylinder with circulation

Now we apply this idea to the flow past a circular cylinder derived in Section 4.1. The potential for a uniform stream in the x-direction is $\phi = Ux = Ur\cos\theta$, and so the total potential for the steady flow past a cylinder of radius R is

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta \tag{80}$$

(cylinder fixed at origin, flow speed U from left to right).

Suppose now that there is circulation round the cylinder, by adding a vortex rotating in the *clockwise* direction:

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta. \tag{81}$$

In the problem without circulation (cf. Fig.33), the stagnation points $\mathbf{u} = (0,0)$ are at the front and the back of the cylinder, i.e. at $\theta = 0$ and $\theta = \pi$ along the surface $r = R$. To find the stagnation points on $r = R$ in the general case (81), we have to look for solutions of $u_\theta = 0$ ($u_r = 0$ by construction):

$$0 = u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{R^2}{R^2}\right)\sin\theta - \frac{\Gamma}{2\pi R}.$$

Thus stagnation points are at angles θ with

$$\sin\theta = -\frac{\Gamma}{4\pi UR}. \tag{82}$$

Equation (82) has two roots in $0 < \theta < 2\pi$ (and hence two stagnation points) if $\Gamma/UR < 4\pi$ (see Fig.36, left). If $\Gamma/UR = 4\pi$, the two stagnation points coincide at $\theta = \frac{3}{2}\pi$

(middle). On the other hand, if $\Gamma/UR > 4\pi$ there are no stagnation points on the surface (right); rather, the stagnation point has moved into the fluid.

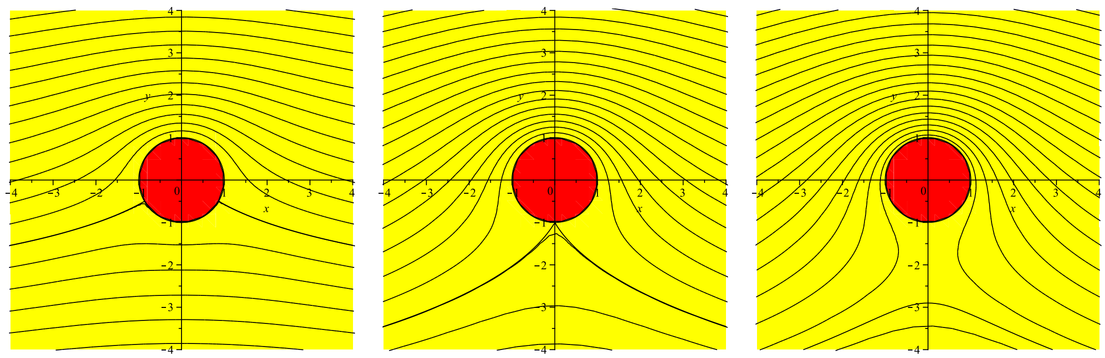


Figure 36: The streamlines around a circular cylinder with circulation $\Gamma < 4\pi UR$, $\Gamma = 4\pi UR$, and $\Gamma > 4\pi UR$.

To calculate the <u>force</u> on the cylinder, we use the unsteady Bernoulli equation,

$$p + \tfrac{1}{2}\rho\mathbf{u}^2 = C,$$

noting that the flow is steady ($C(t) = C$, $\partial\phi/\partial t = 0$). With $p \to p_{atm}$, $\mathbf{u}^2 \to U^2$ as $r \to \infty$,

$$p = p_{atm} + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho\mathbf{u}^2.$$

On the surface of the cylinder,

$$u_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi R} \tag{83}$$

and with $\mathbf{n} = (\cos\theta, \sin\theta)$, the total force is

$$\mathbf{F} = -\int_0^{2\pi} p\mathbf{n}R\,\mathrm{d}\theta.$$

It follows that $\mathbf{F}\cdot\hat{\mathbf{r}} = F_x = 0$, so there is no drag force, as before. However, the transverse force (or lift) is now

$$\begin{aligned} F_y = \mathbf{F}\cdot\hat{\mathbf{y}} &= -\int_0^{2\pi} \left[p_0 + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho(2U\sin\theta + \Gamma/2\pi R)^2\right]\sin\theta R\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \tfrac{1}{2}\rho 4U\sin\theta(\Gamma/2\pi R)\sin\theta R\,\mathrm{d}\theta = \rho U\Gamma. \end{aligned} \tag{84}$$

Hence there is an *upward* lift force on the cylinder (Magnus Effect - see Annalen der Physik **164**, 129 (1853)). This can explain the forces acting on spinning objects (for example footballs or ping-pong balls). A quantitative analysis of the resulting trajectories has been performed by Dupeux et al., New Journal of Physics **12** 093004 (2010).

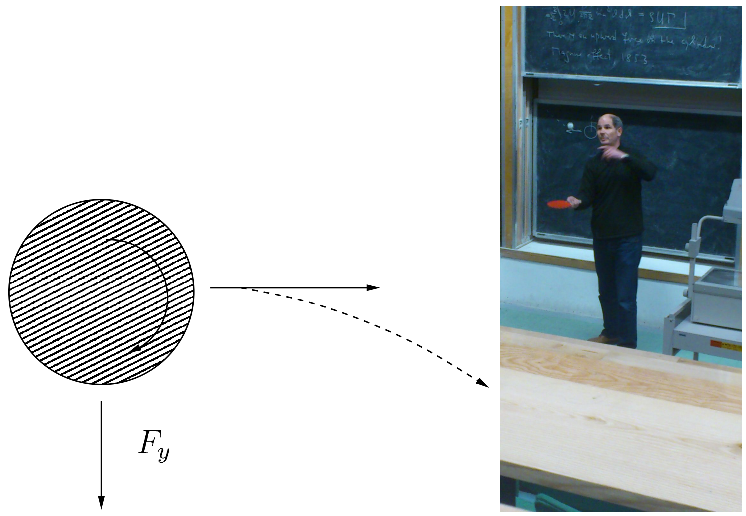


Figure 37: On the left, an illustration of the direction of the Magnus force. On the right, an application to table tennis.

Note that the above calculation refers to cylinder at rest, with a flow past it. It is equivalent to a cylinder moving *to the left* in a stationary medium. Thus a cylinder moving to the right will experience a downward force and deviate in the direction of rotation, which is fairly intuitive, as shown in the Figure. Dupeux et al. investigate the effect more quantitatively, and calculate the resulting spiral flight path. Among others, they show that the Magnus effect is responsible for a famous goal by the Brazilian player Roberto Carlos, see Fig. 38.

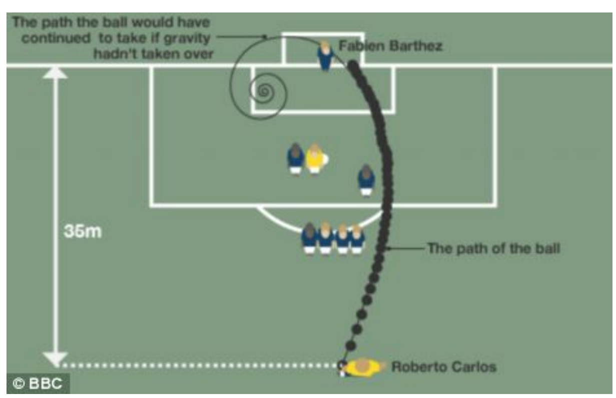


Figure 38: A spinning football describes an increasingly tight spiral.

The effect is also closely related to the lift forces on airplane wings, to which we will come back in more detail. The big question in that case of course is what sets the value of Γ!

4.4 The complex potential

We now introduce the formulation of potential flow using a complex formulation, where each point in the $x-y$ plane is represented by a complex number z. Only this formulation reveals the full power of two-dimensional methods. The first remarkable result is that the potential and the streamfunction are simply the real and imaginary part of the same *complex potential*.

Definition 4.4.1 (Complex potential)
Let $\mathbf{u} = (u, v, 0)$ *be an irrotational, incompressible flow, and* $z = x + iy$. *The the* *complex potential* w *is defined as*

$$w(z) = \phi(x, y) + i\psi(x, y). \tag{85}$$

Theorem 4.4.2 (Complex potential)
The complex potential is an analytic function.

. **Proof:** We have that $\mathbf{u} = \boldsymbol{\nabla}\phi$, and the stream function ψ satisfies (17). Thus

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \end{aligned}\right\} \tag{86}$$

But the equations (86) are the Cauchy-Riemann equations for $w = \phi + i\psi$, and so w is analytic. In particular, if $w(z) = w(x, y)$ is analytic then

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y},$$

i.e. $w'(z)$ is independent of the direction of the derivative.

Properties of the complex potential:

(i) $\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla}^2\psi = 0$ follow directly from (86).

(ii) $\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi = \dfrac{\partial \phi}{\partial x}\dfrac{\partial \psi}{\partial x} + \dfrac{\partial \phi}{\partial y}\dfrac{\partial \psi}{\partial y} = 0$ meaning equipotential curves (where $\phi = const$) are *perpendicular* to streamlines, (where $\psi = const$).

(iii) $\dfrac{\mathrm{d}w}{\mathrm{d}z} = u - iv = q\mathrm{e}^{-i\chi}$ where $q = (u^2 + v^2)^{1/2}$ is the speed of the flow, χ is the angle the flow makes to the x-axis; $\dfrac{\mathrm{d}w}{\mathrm{d}z}$ is known as the *complex velocity* (note the minus sign)!

Example 4.4.3 (Stagnation point (straining) flow)
According to (60), $\psi = 2kxy$, *where* k *is a constant.*

Then

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} = 2kx = \frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \psi}{\partial x} = -2ky = \frac{\partial \phi}{\partial y} \end{aligned}\right\}, \qquad \Rightarrow \phi = k(x^2 - y^2)$$

Hence $w(z) = \phi + i\psi = k(x^2 - y^2) + i2kxy = kz^2$ is the complex potential.

Example 4.4.4 (Line source)
In three dimensions, a two-dimensional source is a line of sources pushing fluid radially outward. According to (21), the flow field is

$$\mathbf{u} = \frac{m}{2\pi r}\hat{\mathbf{r}}.$$

Now the potential and stream functions are

$$\phi = \frac{m}{2\pi}\log r, \quad \psi = \frac{m}{2\pi}\theta. \tag{87}$$

It is recognised immediately that these are the real and imaginary parts of the complex potential

$$w = \frac{m}{2\pi}\log z,$$

since in polars $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$ and so $\log(z) = \log(r) + \log(e^{i\theta}) = \log(r) + i\theta$.

Example 4.4.5 (Line vortex)

We have seen in Section 4.2, that a vortex has the flow field

$$\mathbf{u} = \frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}},$$

and the potential is (cf. (79))

$$\phi = \frac{\Gamma}{2\pi}\theta.$$

But this is the same as the *streamfunction* (87) of a source! Thus we know directly that the complex potential of a vortex must be the same as that of a source, up to a complex rotation:

$$w = -\frac{i\Gamma}{2\pi}\log z.$$

In particular, we can conclude without further calculation that the stream function is

$$\psi = Im\{w\} = -\frac{\Gamma}{2\pi}\ln r.$$

This is a simple example for the application of a powerful theorem from complex analysis: once one knows either the real or the imaginary part of an analytic function, the whole complex function is determined completely. In three dimensions, this will be a line vortex which is completely straight and perpendicular to the xy plane. In general, line vortices are curved, but we disregard this effect here.

Example 4.4.6 (Flow around a cylinder)

We reconsider the problem treated in Section 4.1, but using complex notation.

We have seen that the flow can be described as a superposition of a uniform stream and a dipole. First, a uniform stream in the x-direction has the complex potential

$$w(z) = Uz,$$

since the complex velocity is $u - iv = U$. Second, a dipole is the derivative of a source, which has the potential $w \propto \ln z$. Thus, we arrive at

$$w(z) = Uz - \frac{\mu}{2\pi z}$$

for the potential flow around a cylinder. As before, we want to adjust μ such that the boundary conditions on the surface $|z| = R$ of the cylinder are satisfied. Namely, the stream function

$$\psi = \Im\{w\} = Uy + \frac{\mu y}{2\pi |z|^2}$$

must be a constant for $|z| = R$. Indeed, if we choose $\mu = -2\pi R^2 U$, $\psi = 0$ on the surface. In conclusion, the required complex potential is

$$w(z) = U\left(z + \frac{R^2}{z}\right). \tag{88}$$

This agrees with the flow field (76) found earlier; namely

$$u - iv = \frac{dw}{dz} = U - \frac{UR^2}{z^2} = U - \frac{UR^2}{|z|^4}\left(x^2 - y^2 - 2ixy\right),$$

so that

$$u = U + \frac{UR^2}{r^4}\left(y^2 - x^2\right), \quad v = -\frac{2UR^2}{r^4}xy.$$

On the other hand, since $\mathbf{U}\cdot\hat{\mathbf{x}} = U$ and $\mathbf{n}\cdot\hat{\mathbf{x}} = x/r$, (76) is equivalent to

$$u = U + \frac{UR^2}{r^2}\left(1 - 2\frac{x^2}{r^2}\right) = U + \frac{UR^2}{r^4}\left(x^2 + y^2 - x^2\right)$$

and

$$v = \frac{UR^2}{r^2}\left(-2\frac{xy}{r^2}\right).$$

4.5 Interaction of vortices

Figure 39: A cyclone (point vortex) in the atmosphere.

A line vortex intersects a plane in one point, in two dimensions one can thus describe the vortex by a point (the position $z_0 = x_0 + iy_0$ of the vortex, and its strength Γ. According to Kelvin's theorem, the circulation is conserved, and thus the strength of the vortex is constant in time. The only thing we need to keep track of is the position of the vortex.

In complex notation, the velocity field generated by a vortex located at z_0 is

$$u - iv = -\frac{i\Gamma}{2\pi}\frac{d\log(z - z_0)}{dz} = -\frac{i\Gamma}{2\pi}\frac{1}{z - z_0} = -\frac{i\Gamma}{2\pi}\frac{\overline{z} - \overline{z_0}}{|z - z_0|^2}$$

Now each vortex moves in the velocity field generated by all the other vortices, and so

$$\frac{d\overline{z_1}}{dt} = u - iv|_{allothervortices}$$

In the simplest case of two vortices of strength Γ_1 and Γ_2, located at z_1 and z_2, respectively, the equations of motion are

$$\frac{dz_1}{dt} = \frac{i\Gamma_2}{2\pi} \frac{z_1 - z_2}{|z_1 - z_2|^2}, \quad \frac{dz_2}{dt} = \frac{i\Gamma_1}{2\pi} \frac{z_2 - z_1}{|z_1 - z_2|^2}. \tag{89}$$

Decomposing into real and imaginary parts, the first equation (89) is

$$\frac{dx_1}{dt} = -\frac{\Gamma_2}{2\pi} \frac{y_1 - y_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \quad \frac{dy_1}{dt} = \frac{\Gamma_2}{2\pi} \frac{x_1 - x_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

and correspondingly for the other vortex. The generalisation for N vortices is obvious and involves the sum over all the vortices except for the one that is being convected. How does the motion of two vortices look like ? Let us look at the two possible cases:

4.5.1 Two corotating vortices

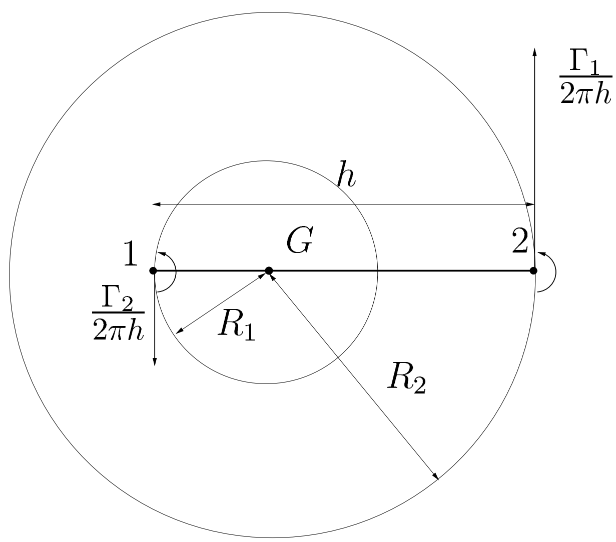


Figure 40: Two vortices with the same sense of circulation rotate around a common center, turning in opposite directions.

The vortex strengths are Γ_1 and Γ_2, both being positive (the two vortices rotate counterclockwise). Let the distance between the two vortices be h. The speeds of the two vortices are then

$$U_1 = \frac{\Gamma_2}{2\pi h}, \quad U_2 = \frac{\Gamma_1}{2\pi h},$$

respectively. From the construction it is clear that at each instant, the vortices move in a direction perpendicular to the line connecting them. This means they have to go around in circles, whose centre G is along the line connecting the vortices. The point G

is stationary in space. The some of the radii R_1, R_2 must satisfy $R_1 + R_2 = h$. Since both vortices make a full turn in the same time, we must have

$$\frac{R_1}{R_2} = \frac{U_1}{U_2} = \frac{\Gamma_2}{\Gamma_1}.$$

Thus for example,

$$R_1 = \frac{\Gamma_2 h}{\Gamma_1 + \Gamma_2}.$$

For two equal vortices of strength $\Gamma_1 = \Gamma_2 = \Gamma$, it follows that $R_1 = R_2 = R$. This means the vortices are always at the opposing sides of a circle of radius R. The time to complete one cycle (the period) is

$$T = \frac{2\pi R}{U} = \frac{8\pi^2 R^2}{\Gamma},$$

and so the whole configuration rotates at an angular velocity of

$$\Omega = \frac{2\pi}{T} = \frac{\Gamma}{4\pi R^2}.$$

4.5.2 Two counterrotating vortices

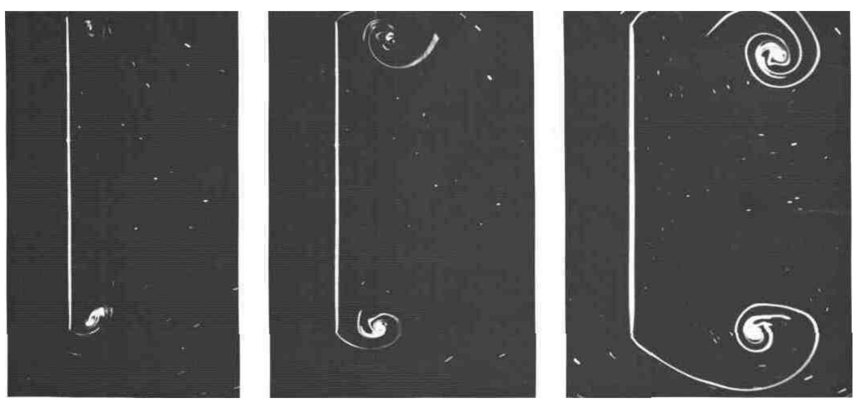


Figure 41: Two counterrotating vortices are created by moving a paddle in water; they travel in the same direction.

Now consider the case that the two vortices have *opposite* sign. This situation is produced easily by the stroke of a paddle, see Fig. 41. In fact since the total vorticity of the system should remain zero according to Kelvin's theorem, typically $\Gamma_1 = -\Gamma_2$. After the paddle is withdrawn, the vortices continue to move according to the other vortex' velocity field. Let the distance between the two vortices again be h.

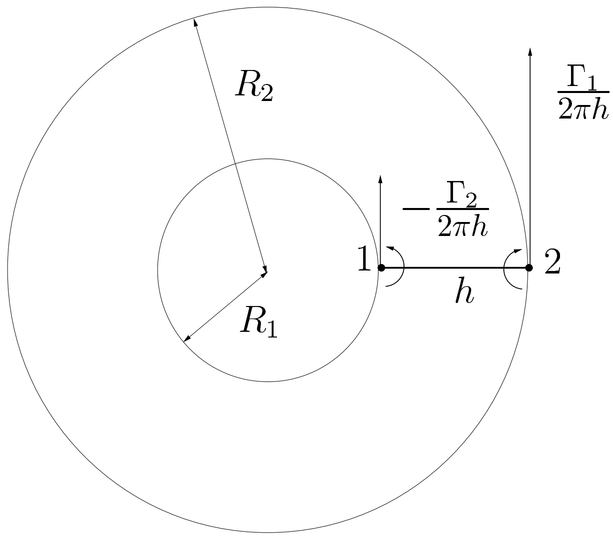


Figure 42: Two vortices with the opposite sense of circulation rotate around a common center, turning in the same direction.

The vortices now move in the same direction, and rotate about the same centre. The radii of this circular motion obeys $R_1/R_2 = -\Gamma_2/\Gamma_1$, noting that the circulations have opposite signs. We now have $R_2 - R_1 = h$, and thus $R_1 = -\dfrac{\Gamma_2 h}{\Gamma_1 + \Gamma_2}$. Thus when the vortices are equal and opposite ($\Gamma = \Gamma_1 = -\Gamma_2$), the radii of both circles tend to infinity. In this case the vortices move in a straight line, and *translate* uniformly at speed $U = \dfrac{\Gamma}{2\pi h}$. This is what's seen in the experimental pictures of two vortices being created behind a paddle.

4.6 The method of images

4.6.1 A vortex next to a wall

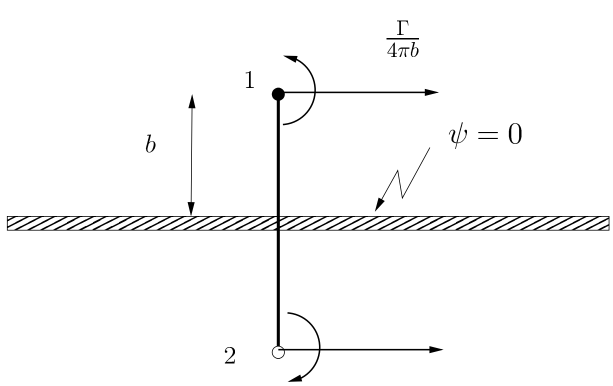


Figure 43: A vortex next to a wall moves parallel to it.

Consider the motion of a vortex located at some position $z_0 = a + ib$ in the upper half of the complex plane. The x axis is the location of a solid wall. Thus we have to satisfy the boundary condition of no flow through the wall: $v = 0$ for $y = 0$. Simply taking the flow field of the vortex,

$$u - iv = -\frac{i\Gamma}{2\pi} \frac{x - a - i(y - b)}{(x - a)^2 + (y - b)^2}, \tag{90}$$

does not satisfy this condition: the flow field must be modified by the presence of the wall.

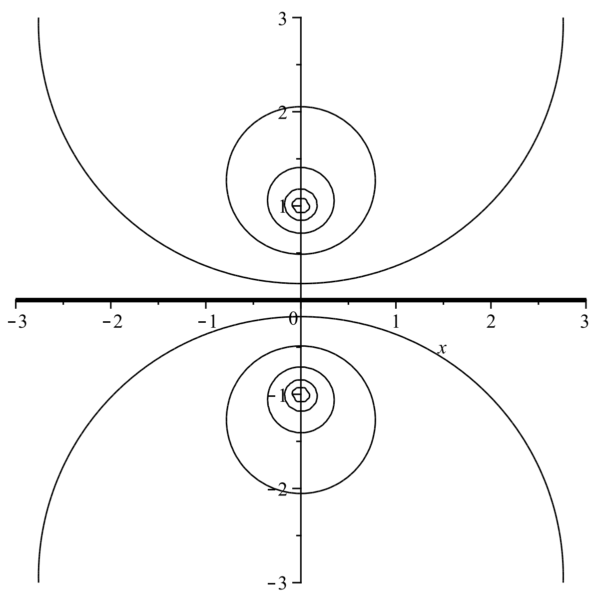


Figure 44: Streamlines of a vortex next to a wall.

The method of images is a technique that permits to find solutions to Laplace's equation which satisfy the right boundary condition on the solid wall. Near the vortex at z_0, the solution should look like (90). The method of solution is indicated in the figure: imagine replacing the wall by another *image* vortex that is placed at an equal distance on the other side of the wall. Its position is where an image in a mirror would appear. And equally like a mirror image, the sense of rotation of the image vortex is reversed. By symmetry it is clear that the v component of the image vortex has opposite sign, so on the line of symmetry (the locus of the wall), the two contributions cancel and $v = 0$. The boundary condition on the wall is satisfied and we have solved our problem! In other words, the flow problem we want to solve (*one* vortex in the presence of a wall) is exactly the same as the flow problem of two vortices (the original vortex and its image) *without* the wall.

Let's check if our reasoning was correct. It is easiest to do the calculation using complex notation. If the vortex is at z_0, its image is at $\overline{z_0}$. Since the image also has the opposite sense of rotation, there is a minus sign in front of it. The total potential (vortex + image) becomes

$$w(z) = -\frac{i\Gamma}{2\pi} \log(z - z_0) + \frac{i\Gamma}{2\pi} \log(z - \overline{z_0}). \tag{91}$$

We have to verify that $Im\{w\} = \psi = const$ for $z = x$ real! Now since $\overline{\ln(z)} = \ln(\overline{z})$ (why?) we have

$$\overline{w(z)} = \frac{i\Gamma}{2\pi} \log(\overline{z} - \overline{z_0}) - \frac{i\Gamma}{2\pi} \log(\overline{z} - z_0).$$

But if z is real, $z = \overline{z}$, and thus $w(x) - \overline{w(x)} = 0$, and so $\psi = 0$ along the wall.

In fact, we have shown that the problem of a vortex near a wall is mathematically equivalent to the problem of two counterrotating vortices of equal strength, which in the previous section we showed to move at constant speed in the direction perpendicular to the line connecting them, in other words, parallel to the wall. Let us check this fact as well, using complex notation. The vortex in question moves in the flow field of its image, i.e.

$$u - iv = \frac{i\Gamma}{2\pi} \frac{x - a - i(y + b)}{(x - a)^2 + (y + b)^2},$$

evaluated at $z = a + ib$. Thus

$$u - iv = \frac{i\Gamma}{2\pi} \left(\frac{-2ib}{4b^2} \right) = \frac{\Gamma}{4\pi b},$$

and the vortex moves with a horizontal speed of $\dfrac{\Gamma}{4\pi b}$, which corresponds to the earlier result with $h = 2b$.

This leads us to the following result, which permits to find the flow generated by *any* combination of singularities next to a wall. Let the wall be the real axis of the complex plain, and let $f(z)$ describe all singularities (vortices, sources, etc.) present in the flow domain $y > 0$. For example, $f(z) = \dfrac{i\Gamma}{2\pi} \ln(z - z_0)$ for a vortex and $f(z) = -\dfrac{\mu}{2\pi(z - z_0)}$ for a dipole.

Theorem 4.6.1 (Method of images)

Let $w(z) = f(z)$ would be the complex potential of the flow without the wall. Then

$$w(z) = f(z) + \overline{f(\overline{z})} \tag{92}$$

is the potential in the presence of the wall.

Note that in the the second member of (92), we take the complex conjugate of everything *except* of z itself.

Proof: We have to show that $\psi = Im\{w(z)\} = 0$ if $z = x$ is real. But if this is the case then $\overline{z} = z$, and so

$$w(x) - \overline{w(x)} = f(x) + \overline{f(x)} - (\overline{f(x)} + f(x)) = 0,$$

which proves the theorem.

Example 4.6.2 (Source next to a wall)

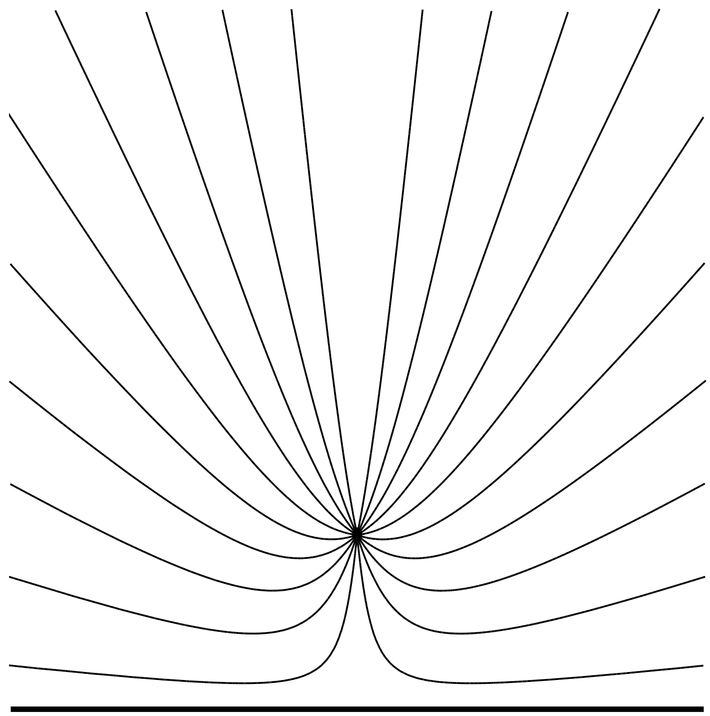


Figure 45: The streamlines of a source next to a wall.

Consider a source of strength m *placed at* $z = ib$ *above a wall. Then* $f(z) = \frac{m}{2\pi}\ln(z - ib)$ *and* $\overline{f(\overline{z})} = \frac{m}{2\pi}\ln(z + ib)$*. Thus the solution we are seeking is*

$$w(z) = \frac{m}{2\pi}\left(\ln(z - ib) + \ln(z + ib)\right) = \frac{m}{2\pi}\ln(z^2 + b^2). \tag{93}$$

The streamfunction is $\psi = \Im\{w\}$*, whose level lines are the streamlines, plotted in Fig. 45.*

4.6.2 A vortex pair

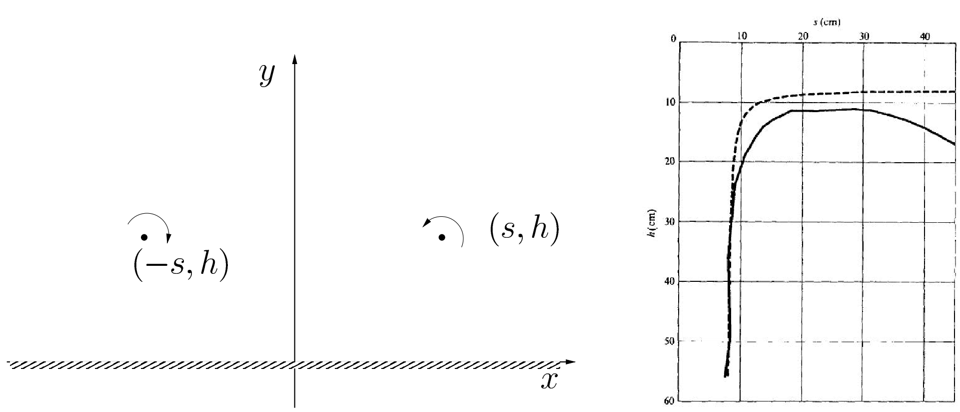


Figure 46: Two counterrotating vortices next to a wall. On the left, the configuration; on the right the experimentally recorded trajectory of one of the vortices.

A pair of counter-rotating vortices of strength Γ is produced by the wings of a plane moving on a runway. To determine the flow field we have to use the method of images;

the the position of the vortices be (s, h) and $(-s, h)$, as shown in Fig. 46. Let $z_0 = s + ih$ be the position of the right vortex; then $-\overline{z}_0 = -s + ih$ is the position of the vortex on the left. Now the complex potential of the vortices in the upper half plane is

$$w = \frac{i\Gamma}{2\pi}\left(-\ln(z - z_0) + \ln(z + \overline{z}_0)\right) \equiv f(z).$$

According to our recipe, the total potential in the presence of the wall at $y = 0$ is

$$w = f(z) + \overline{f(\overline{z})} = \frac{i\Gamma}{2\pi}\left(-\ln(z - z_0) + \ln(z + \overline{z}_0) + \ln(z - \overline{z}_0) - \ln(z + z_0)\right).$$

The velocity at z_0 due to the other three vortices is $\left.\frac{w_3}{dz}\right|_{z=z_0}$, where

$$w_3 = \frac{i\Gamma}{2\pi}\left(\ln(z + \overline{z}_0) + \ln(z - \overline{z}_0) - \ln(z + z_0)\right).$$

Thus the velocity at z_0 is

$$u - iv = \frac{i\Gamma}{2\pi}\left(\frac{1}{z_0 + \overline{z}_0} + \frac{1}{z_0 - \overline{z}_0} - \frac{1}{2z_0}\right) = \frac{i\Gamma}{2\pi}\left(\frac{1}{2s} + \frac{1}{2ih} - \frac{1}{2(s + ih)}\right)$$
$$= \frac{\Gamma}{4\pi}\left[\frac{s^2}{h(s^2 + h^2)} + i\frac{h^2}{s(s^2 + h^2)}\right].$$

Thus the equations of motion for the vortex at (s, h) are

$$\dot{s} = \frac{\Gamma}{4\pi}\frac{s^2}{h(s^2 + h^2)}, \qquad \dot{h} = -\frac{\Gamma}{4\pi}\frac{h^2}{s(s^2 + h^2)}. \tag{94}$$

Solving the equations is left as an exercise.

4.7 Mappings and transformations

The most powerful tool of complex analysis consists in mapping the flow domain into another. As soon as I have found a mapping that is conformal, I have transformed the problem from one geometry to another.

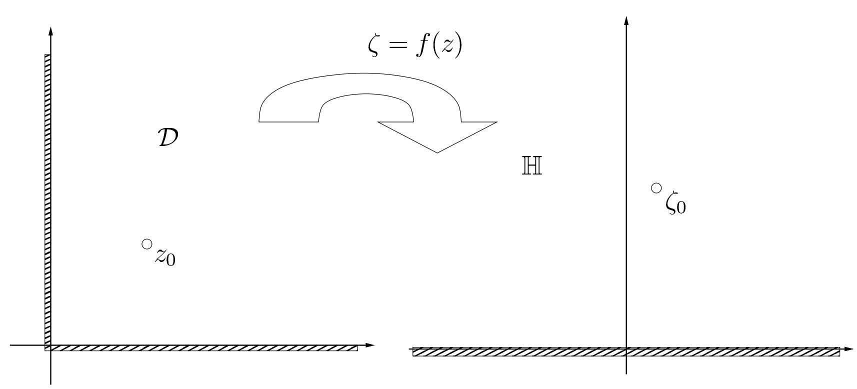


Figure 47: Via a complex mapping $f(z)$ a corner can be mapped into the upper half plane.

Imagine we want to solve a flow problem in a given domain $\mathcal{D}$. This means we are looking for a function $w(z)$ (the complex potential) with the following properties:

- $w(z)$ is analytic in the domain $\mathcal{D}$, except perhaps for one or more singularities at z_0 (and other places).
- $Im\{w(z)\}$ = const on the boundaries of the domain.

To find such a complex potential, it is enough to find a mapping into a new domain $\mathcal{D}_1$ such that the flow problem can be solved in $\mathcal{D}_1$. We have the following:

Theorem 4.7.1 (Complex mapping)
Let $\zeta = f(z)$ be a mapping that is analytic everywhere in $\mathcal{D}$, and which maps onto a new domain $\mathcal{D}_1$. We seek a complex potential $w(z)$ with prescribed singularities $z_0 \in \mathcal{D}$, and boundary condition $Im\{w(z)\} = const$ on $\partial\mathcal{D}$.

Let the singularity at z_0 be mapped to a new place $\zeta_0 = f(z_0)$ and the boundary of $\mathcal{D}$ is mapped to the boundary of $\mathcal{D}_1$. Let $w_1(\zeta)$ be the solution to the flow problem in $\mathcal{D}_1$ (which has the singularity corresponding to z_0 at ζ_0). Then

$$w(z) = w_1(f(z)) \tag{95}$$

is a solution to the original flow problem.

Proof: Since $w_1(\zeta)$ is analytic in $\mathcal{D}_1$ (apart from singularities), $w(z)$ is analytic in $\mathcal{D}$; at z_0 it has the prescribed singularity. This is the first requirement. In addition, $Im\{w(\zeta)\}$ is constant on the boundary of $\mathcal{D}_1$. But this means that $Im\{w(z)\} = Im\{w_1(f(z))\}$ is also constant, which meets the second requirement, and proves the theorem.

Example 4.7.2 (A source in a corner)

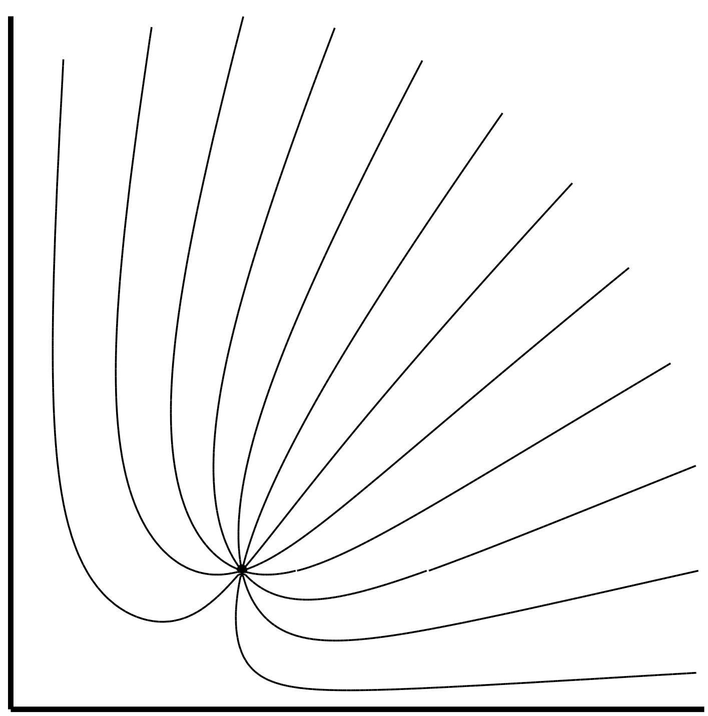


Figure 48: The flowlines of a source in a corner, given by the complex potential (96).

The function $\zeta = f(z) \equiv z^2$, $z = \zeta^{1/2}$ maps the first quadrant onto the half-plane. Namely, if $z = re^{i\theta}$, then $\zeta = r^2e^{2i\theta} = r^2e^{i\phi}$. Thus the domain $0 \le \theta < \pi/2$ is mapped onto $0 \le \phi < \pi$. The source at z_0 maps to a source at $\zeta_0 = z_0^2$. Now the solution in the ζ-domain is

$$w_1(\zeta) = \frac{m}{2\pi}\log(\zeta-\zeta_0) + \frac{m}{2\pi}\log(\zeta-\overline{\zeta}_0)$$

as we have seen before, using the method of images. So

$$\begin{aligned} w(z) = w_1(z^2) &= \frac{m}{2\pi}\left(\log(z^2-z_0^2)+\log(z^2-\overline{z}_0^2)\right) \\ &= \frac{m}{2\pi}\left(\log(z-z_0)+\log(z+z_0)+\log(z-\overline{z}_0)+\log(z+\overline{z}_0)\right) \end{aligned} \tag{96}$$

Thus the result can be written as a superposition of four sources of equal strength, located at $z_0, -z_0, \overline{z}_0$, and $-\overline{z}_0$. The same result could have been obtained (albeit a bit more laboriously), using the method of images. First, we mirror at the x-axis, then at the y-axis. Fig. 48 is for the case $z_0 = 2+i$.

Example 4.7.3 (Rotation)

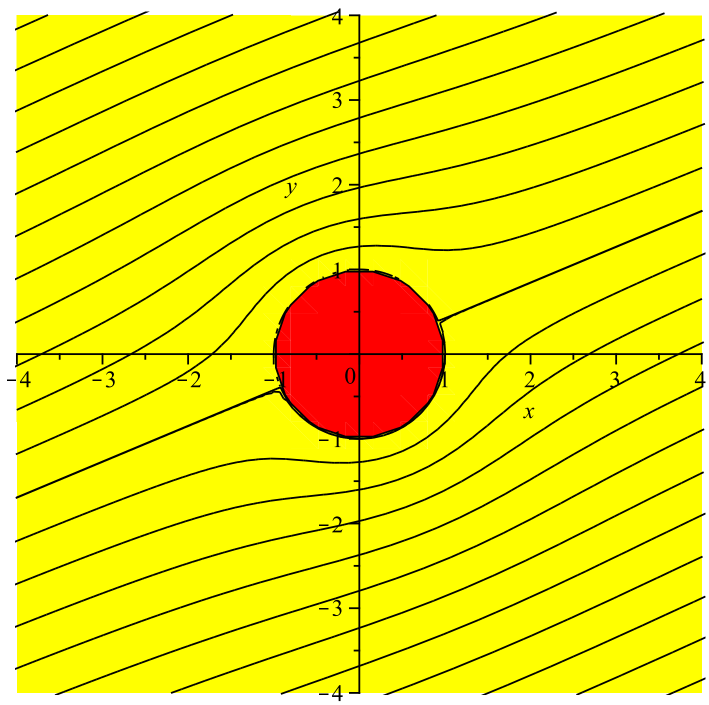


Figure 49: The flowlines of a uniform flow around a cylinder, which arrives under an angle α.

Consider any flow problem, for example the flow around a cylinder, given by (88):

$$w_1(\zeta) = U\left(\zeta + \frac{R^2}{\zeta}\right).$$

Let $\zeta = z\mathrm{e}^{-i\alpha} \equiv f(z)$, *which is a rotation of the axes by* α *in the anticlockwise direction. Namely, for* $z = \mathrm{r}\mathrm{e}^{i\theta}$, $\zeta = \mathrm{r}\mathrm{e}^{i(\theta-\alpha)}$.

Then in the z-plane,

$$w(z) = w_1(ze^{-i\alpha}) = U\left(z\mathrm{e}^{-i\alpha} + \frac{R^2}{z}\mathrm{e}^{i\alpha}\right). \tag{97}$$

The flowlines (equipotential lines of $\psi = \Im\{w\}$) are plotted in Fig. 49.

Note 4.7.4 (Calculation of velocities)
Velocities are most easily found from

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv = \frac{\mathrm{d}w_1}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}$$

E.g. in above $u - iv = (u_1 - iv_1)\mathrm{e}^{-i\alpha}$.

4.8 The Joukowski mapping: circles to ellipses

A particularly useful application of the mapping idea concerns the flow around bodies. We have solved the problem of the flow around a cylinder. Thus if we can find a conformal mapping between the unit circle and any given shape, we have solve the flow problem around this shape.

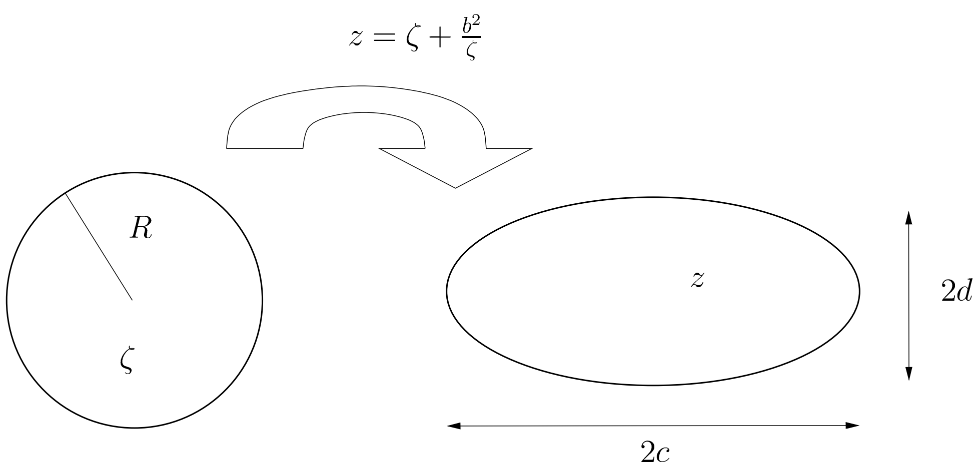


Figure 50: The Joukowski mapping (98) maps a circle to an ellipse.

Consider the (inverse) mapping

$$z = \zeta + \frac{b^2}{\zeta}, \tag{98}$$

called the Joukowski mapping, and illustrated in Fig. 50. Consider a circle of radius R, whose surface in the ζ-plane is described by the polar representation $\zeta = R\mathrm{e}^{i\theta}$. Under the Joukowski mapping,

$$z = R\mathrm{e}^{i\theta} + \frac{b^2}{R}\mathrm{e}^{-i\theta} = \left(R + \frac{b^2}{R}\right)\cos\theta + i\left(R - \frac{b^2}{R}\right)\sin\theta = c\cos\theta + id\sin\theta$$

is an ellipse with axes length $2c$ and $2d$.

The semiaxes come out to be

$$c = R + \frac{b^2}{R}, \quad d = R - \frac{b^2}{R}. \tag{99}$$

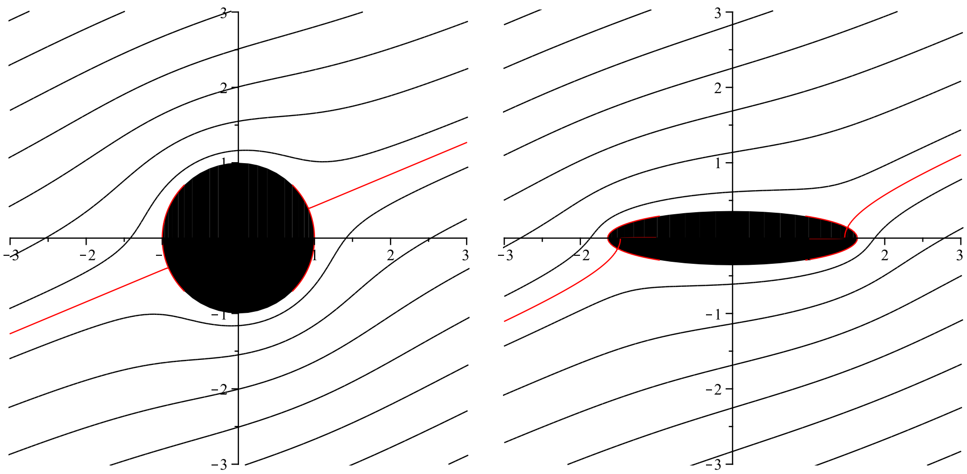


Figure 51: The flowlines areound a circle are mapped to an ellipse by the Joukowski mapping (98).

Now consider the uniform flow past an ellipse. To model an arbitrary angle α between the direction of the flow and the semi-major axis, we consider the flow around a cylinder that approaches the x-axis under an angle α:

$$w_1(\zeta) = U\left(\zeta \mathrm{e}^{-i\alpha} + \frac{R^2}{\zeta}\mathrm{e}^{i\alpha}\right).$$

In principle, one can find $w(z)$ using the inverse of the Joukowski mapping

$$\zeta = f(z) = \frac{1}{2}(z + \sqrt{z^2 - 4b^2}),$$

so that $w(z) = w_1(f(z))$.

However, the resulting expressions are often not so useful. For example, to find the streamlines, it is much easier to find the streamlines of the $w_1(\zeta)$ in the ζ-plane, and then to transform them using (98). This is how the Fig.51 was produced. If one wants to calculate the velocity, one uses

$$u - iv = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}w_1}{\mathrm{d}\zeta}\frac{1}{\mathrm{d}z/\mathrm{d}\zeta} = \frac{U(\mathrm{e}^{-i\alpha} - R^2\mathrm{e}^{i\alpha}/\zeta^2)}{(1 - b^2/\zeta^2)}.$$

On the cylinder, $\zeta = R\mathrm{e}^{i\theta}$, so

$$u - iv = \frac{U(\mathrm{e}^{-i\alpha} - \mathrm{e}^{i\alpha}\mathrm{e}^{-2i\theta})}{(1 - (b^2/R^2)\mathrm{e}^{-2i\theta})} = \frac{2iU\sin(\theta - \alpha)}{(\mathrm{e}^{i\theta} - (b^2/R^2)\mathrm{e}^{-i\theta})}. \tag{100}$$

This means there are stagnation points at $\theta = \alpha$ and $\theta = -\pi + \alpha$. This point is where a streamline leaves the surface. In other words, this streamline (plotted in red) has the same value of the streamfunction ψ then the surface of the ellipse.

4.9 Lift

Now we want to fly! In principle, we know how to construct the flow around a wing of arbitrary shape, we only have to find the transformation, starting from a circle. We have seen already that the key ingredient is to have circulation around the wing. We have calculated the lift in the case of a cylindrical cross-section, but what is it for an arbitrary shape? To answer this question, we have the following theorem, which gives the force on a body of arbitrary shape, using complex notation.

Theorem 4.9.1 (Blasius' theorem)

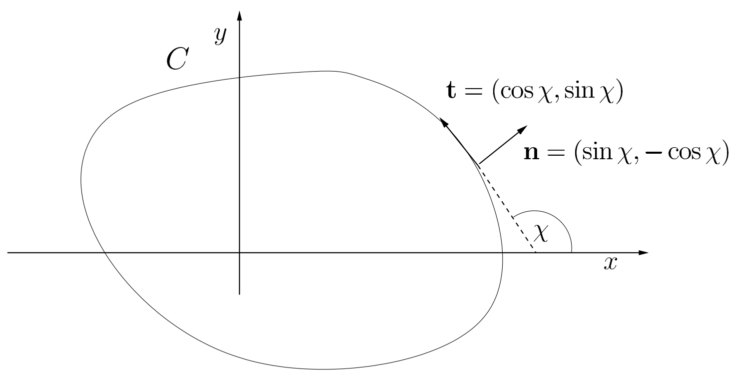


Figure 52: If χ is the angle made by the tangent line with the x-axis, the tangent and normal is given by (103).

Let C be a simple, closed curve, describing a body in a stready stream U. Let the complex potential outside of the body be $w(z)$. Then the force $\mathbf{F} = (F_x, F_y)$ on the body is given by

$$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{101}$$

Proof: The complex velocity is given by

$$\frac{dw}{dz} = u - iv = q\mathrm{e}^{-i\chi}, \tag{102}$$

where $|\mathbf{u}| = q$ is the speed of the flow, and χ is the angle the flow direction makes to the horizontal. So, according to Bernoulli (no gravity):

$$p = p_0 - \tfrac{1}{2}\rho q^2,$$

where $p_0 = p_{atm} + \rho U^2/2$ is a constant; U is the flow speed at infinity.

But on the surface of body, the flow is in the tangential direction. So χ is also the angle the tangent line of the body makes with the real axis, and as shown in the above Figure, we have $\mathbf{t} = (\cos\chi, \sin\chi)$, $\mathbf{n} = (\sin\chi, -\cos\chi)$, for the tangent and normal vectors, respectively. If we interpret both components as real and imaginary parts of a complex number, this is the same as

$$t = e^{i\chi}, \quad n = -ie^{i\chi}, \tag{103}$$

where t and n are complex numbers representing the tangent and the normal.

To compute the total force $\mathbf{F}$ on the body, we have to do the integral

$$\mathbf{F} = -\oint_C p\mathbf{n}\, \mathrm{d}s,$$

where s is the arclength along C. On the other hand, a complex line element along C is

$$dz \equiv dx + idy = \left(\frac{dx}{ds} + i\frac{dy}{ds}\right) ds = t\, \mathrm{d}s = \mathrm{e}^{i\chi}\, \mathrm{d}s.$$

Here we have used that by definition, the tangent vector is

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right),$$

and (103) in the last step.

Now we can convert everything to complex notation, where it proves convenient to define a *complex force* $F = F_x - iF_y$. Then

$$\begin{aligned} F = F_x - iF_y = -\oint_C p\overline{n}\, \mathrm{d}s &= -i\oint_C p\mathrm{e}^{-i\chi}\, \mathrm{d}s \\ &= -i\oint_C p_0(dx - idy) + \frac{i\rho}{2}\oint_C q^2\mathrm{e}^{-i\chi}\, \mathrm{d}s \\ &= \frac{i\rho}{2}\oint_C (q\mathrm{e}^{-i\chi})^2\, \mathrm{d}z = \frac{i\rho}{2}\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z \end{aligned}$$

(the term proportional to p_0 vanishes because the integral of a total differential over a closed loop is zero), and we have used (102) in the last step. This proves Blasius' theorem.

Example 4.9.2 (Cylinder with circulation) *Consider the problem of Section 4.3,*

$$w(z) = Uz + U\frac{R^2}{z} - \frac{i\Gamma}{2\pi}\log z,$$

so

$$\frac{\mathrm{d}w}{\mathrm{d}z} = U - U\frac{R^2}{z^2} - \frac{i\Gamma}{2\pi z}.$$

Applying (101) to the circle C gives

$$\begin{aligned} F_x - iF_y &= \frac{i\rho}{2}\oint_C \left(U - U\frac{R^2}{z^2} - \frac{i\Gamma}{2\pi z}\right)^2 \mathrm{d}z \\ &= \frac{i\rho}{2}\oint_C \left(U^2 - \frac{iU\Gamma}{\pi z} + \text{terms } \{z^{-2}, z^{-3}, z^{-4}\}\right) \mathrm{d}z \end{aligned}$$

Now use Cauchy's residue theorem $\oint_C \frac{\mathrm{d}z}{z} = 2\pi i$, $\oint_C \frac{\mathrm{d}z}{z^n} = 0$, $n \neq 1$ and

$$F_x - iF_y = i\rho U\Gamma,$$

which agrees with (84).

Example (4.9.2) shows that Blasius' theorem yields a much more general result, since only the residue of the integral comes into play. This is the

Theorem 4.9.3 (The Kutta-Joukowski lift theorem) *Consider a body of arbitrary cross-section C, in a uniform stream U in the x-direction, which generates circulation of strength Γ. Then the force on the body is given by:*

$$F_x - iF_y = iU\rho\Gamma. \tag{104}$$

Hence the drag (force in the direction of the stream) on an arbitrary body is zero and the lift force is $F_{lift} = -U\rho\Gamma$. Note that the lift force is by definition the force in the direction normal to the flow.

Proof: Far away from the body,

$$w(z) \to Uz - \frac{i\Gamma}{2\pi}\log z$$

(origin inside C) so

$$\frac{\mathrm{d}w}{\mathrm{d}z} \approx U - \frac{i\Gamma}{2\pi z}, \qquad z \to \infty.$$

Since there are no singularities in the flow between C and $|z| = R$, the integral must be independent of any chosen closed path outside C:

$$\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = \lim_{R\to\infty} \oint_{|z|=R} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z.$$

So

$$\oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = \oint_{|z|=R} \left(U - \frac{i\Gamma}{2\pi z}\right)^2, = -\frac{i\Gamma U}{\pi}\oint_{|z|=R} \frac{1}{z}\,\mathrm{d}z = 2U\Gamma,$$

using the same reasoning as for the cylinder. This proves the theorem.

4.10 Oblique flow past plates

.

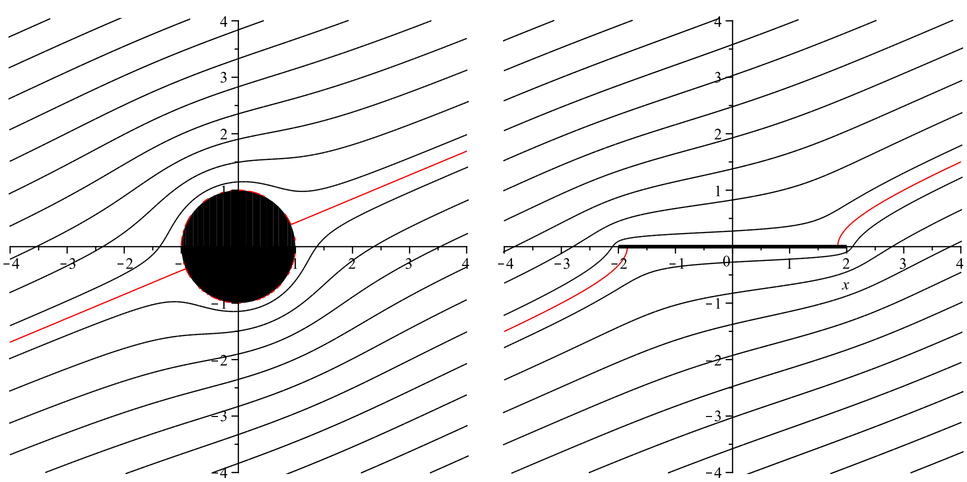


Figure 53: The mapping of a circle to a plate, using (105).

We have developed a wonderful theory for lift. The only problem is that we don't know how to determine the value of Γ which determines the lift. Indeed, the wings of an airplane do not have circular cross section: if cylinder is placed in a uniform stream (left side of the Figure), there is no reason why the flow should turn either way, and thus there is no circulation. The secret is to fashion the shape so as to induce lift. Namely, the tail of wing is shaped to have a sharp corner, so as to force the flow to separate at that point.

As the simplest possible model for such a situation consider the flow around a flat plate, which is placed in a uniform stream at an angle α. A flat plate is obtained from an ellipse by letting $d \to 0$. According to (99), this amounts to putting $b = R$ in the Joukowski transformation (98):

$$z = \zeta + \frac{R^2}{\zeta}. \tag{105}$$

Then the width of the plate is $2c = 4R$. As is seen from the Figure, the flow separates from the plate beyond the trailing edge, so that fluid particles very close to the plate have to go around the sharp corner before leaving the plate.

This situation is reflected by the velocity on the surface of the plate, which is

$$u - iv = \frac{U \sin(\theta - \alpha)}{\sin\theta},$$

putting $b = R$ in (100). Clearly, u diverges as as $\theta \to 0, \pi$, i.e. at the sharp edges of the plate. This is a physically untenable situation; fluid particles are not likely to make such a sharp turn without leaving the plate altogether, especially not at infinite speed! The same conclusion can be drawn from the concept of "adverse pressure gradient", introduced earlier. The speed is very great at the trailing edge and decreases as one moves up the plate. According to Bernoulli, this means that the pressure increases, in other words fluid particles experience an adverse pressure gradient. Instead of following a path of adverse pressure, fluid particles prefer to leave the body, producing a point of separation at the trailing edge. This requirement is known as the **Kutta condition**.

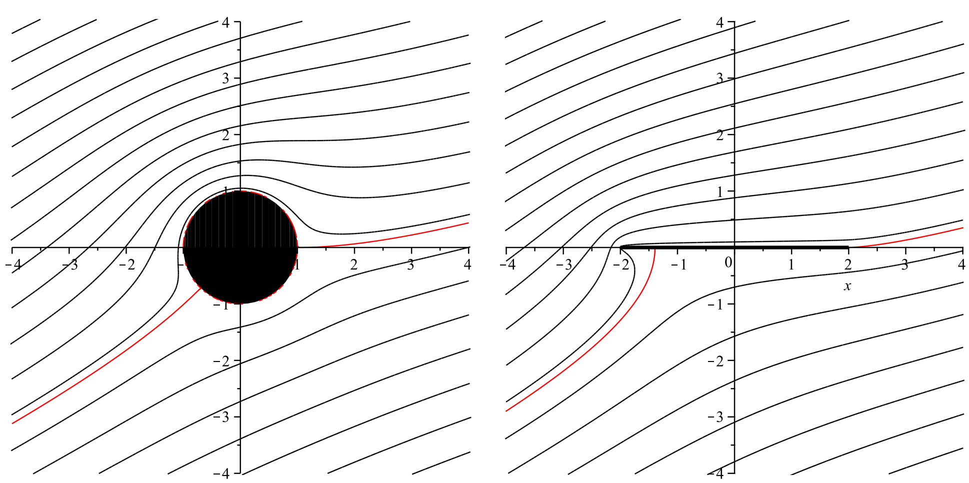


Figure 54: Flow around a plate satisfying the Kutta condition (107).

As we have seen in our calculation of the flow around a cylinder with circulation, the point of separation can be made to move by adding circulation:

$$w_1(\zeta) = U\left(\zeta e^{-i\alpha} + \frac{R^2}{\zeta} e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\log\zeta.$$

Then the complex velocity of the flow around a plate ($b = R$) is

$$u - iv = \frac{dw_1}{d\zeta}\frac{d\zeta}{dz} = \left[U\left(\mathrm{e}^{-i\alpha} - \frac{R^2}{\zeta^2}\mathrm{e}^{i\alpha}\right) - \frac{i\Gamma}{2\pi\zeta}\right]\left(1 - \frac{R^2}{\zeta^2}\right)^{-1}.$$

Putting $\zeta = R\mathrm{e}^{i\theta}$, we find the velocity on the plate:

$$u - iv = \left[U\left(\mathrm{e}^{-i\alpha+i\theta} - \mathrm{e}^{i\alpha-i\theta}\right) - \frac{i\Gamma}{2\pi R}\right]\frac{\mathrm{e}^{-i\theta}}{1-\mathrm{e}^{-2i\theta}} = \left[U\sin(\theta-\alpha) - \frac{\Gamma}{4\pi R}\right]\frac{1}{\sin\theta}. \qquad (106)$$

Let us focus on the trailing edge, corresponding to $\theta \to 0$. (assume for example that the front of the plate is slightly 'rounded', so that separation is not as important there). Thus we want $u - iv$ to be finite for $\theta \to 0$, a requirement which is another incarnation of the Kutta condition. This can be achieved by choosing Γ such that the square bracket on the right of (106) *vanishes* for $\theta = 0$. In other words,

$$\Gamma = -4\pi R U \sin\alpha, \qquad (107)$$

which is the Kutta condition for the plate. This is the value chosen for the above Figure, where the flow is seen to leave the plate smoothly in the direction of the orientation of the plate. Now the singularity cancels and the velocity becomes

$$u - iv = \frac{U}{\sin\theta}(\sin(\theta-\alpha) + \sin\alpha) \to U\cos\alpha, \qquad \text{as } \theta \to 0,$$

a finite value!

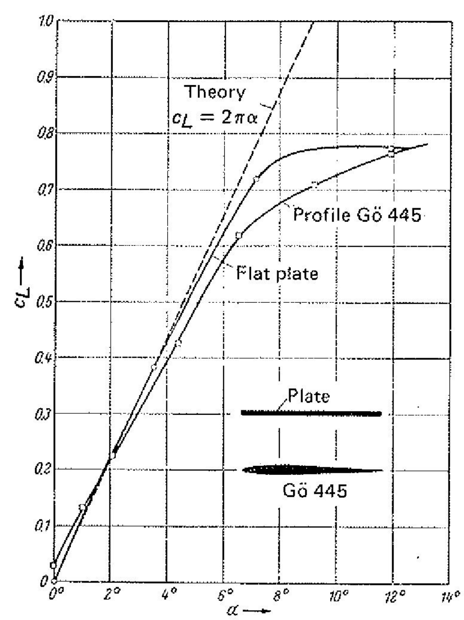


Figure 55: The experimentally measured lift coefficient for a plate, compared to the theoretical result (110).

The wonderful thing is that we have now determined the circulation uniquely, so we can calculate the lift using Blasius' theorem:

$$F_{lift} = -U\rho\Gamma = 4\pi R\rho U^2 \sin\alpha, \qquad (108)$$

where $4R$ is the width of the plate. Once more, note that this is the force acting normal to the flow, so it is at an angle α relative to the orientation of the plate. It is customary to define a lift coefficient c_L by

$$F_{lift} = c_L A_w \frac{\rho U^2}{2}, \tag{109}$$

where $A_w = 4R$ is the area of the wing (per unit length), also called the chord. Thus we have found that for the plate

$$C_L^{(plate)} = 2\pi \sin\alpha \approx 2\pi\alpha \tag{110}$$

In Fig.55, (110) is compared to experiment, and it works really well, as long as the angle of attack α is small. If however α becomes too large the theory fails abruptly. The reason is clear from the two photographs below. As long as α is small, the flow remains laminar and attached to the wing. As α is too great, the flow separates and a completely different type of description must be sought, see Fig. 56.

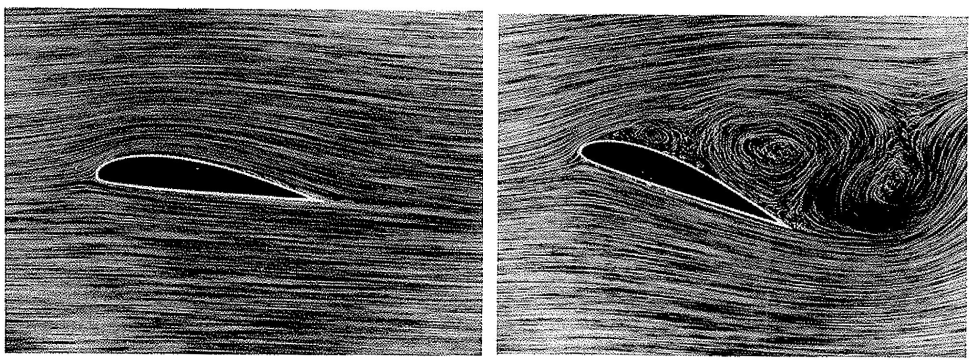


Figure 56: The flow around a wing. At small angles α, the flow separates smoothly. As α becomes large, the flow separates from the wing and vortices are formed.

Finally, we comment on the presence of circulation around the wing, which is the crucial ingredient needed for flying. In the first instance, there is nothing wrong with that from the point of view of a potential flow description. However, where is this circulation coming from when one imagines starting up the flow, with zero circulation? Since the net circulation in a large circle around the wing must vanish initially, and the Kutta condition requires a circulation $\Gamma < 0$ around the wing, this means that in the process of the point of separation moving to the trailing edge, vortices of positive circulation (rotating counterclockwise) are shed from the wing. Eventually the vortices are convected downstream, and no longer matter for the problem.

5 Waves and free surface flows

Free surface flows are in some ways very different from what we have done before. In all problems considered so far, the domain $\mathcal{D}$ in which to solve the problem is given (for example some box or the exterior of an aeroplane wing). A free surface, on the other hand, moves, so the domain varies in time. The key feature, however, is that the domain does not vary according to some predetermined program. Instead, it moves in response to the flow itself, that is in response to the flow solution (which of course itself depends on the domain).

This leads to solutions which are quite different in character to the fixed-boundary solutions we were considering so far (and generally speaking much more interesting solutions)! Another nice feature of free surface flows is that they are easy to observe experimentally, one just needs to track the motion of the free surface.

5.1 Nonlinear free-surface motion

We begin with the general equations for free-surface flow, assuming that the flow inside the fluid is potential, and the fluid is incompressible. We assume for now that the flow is 2D.

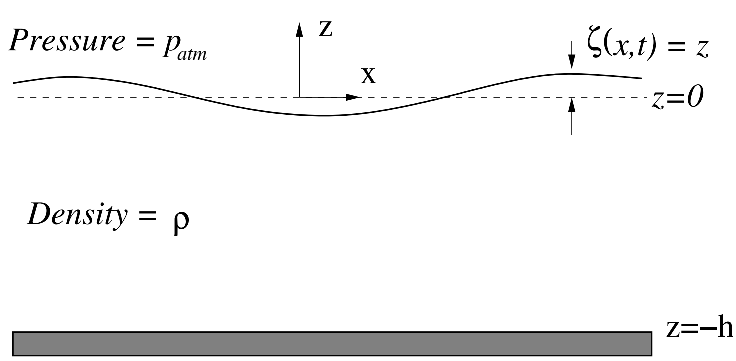


Figure 57: Schematics for wave motion on a fluid layer of thickness h above a solid wall.

Hence the fluid flow is described by

$$\mathbf{u} = \boldsymbol{\nabla}\phi, \quad \boldsymbol{\nabla}^2\phi = 0,$$

where $\mathbf{u} = (u, 0, w)$ and $\phi = \phi(x, z, t)$. We choose $z = 0$ to coincide with the undisturbed free surface, and the bottom of the fluid is at $z = -h$, see Fig. 57. Let the surface of the water in motion be given by $z = \zeta(x, t)$.

The interesting part of the problem are its boundary conditions:

(i) At the lower boundary $z = -h$, the vertical velocity must vanish (kineamtic b.c.):

$$0 = \mathbf{n} \cdot \boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial z} = w.$$

In other words,

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = -h. \tag{111}$$

(ii) On $z = \zeta(x,t)$, the kinematic boundary condition on a moving surface is $\dfrac{DS}{Dt} = 0$, where the free surface is the zero set of $S(\mathbf{r}, t)$. If the surface can be represented by a height function $\zeta(x,t)$ (i.e. we are not allowed overhangs) we can define a function

$$S(x,z,t) = z - \zeta(x,t)$$

which is indeed zero at the free surface. Thus

$$0 = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)(z - \zeta(x,t)) = -\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial z} \qquad \text{on } z = \zeta,$$

so the kinematic boundary condition becomes

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z}, \qquad \text{on } z = \zeta. \tag{112}$$

(iii) The dynamic boundary condition at the free surface is that the pressure equals the exterior athmospheric pressure: $p = p_{atm}$ (const). on $z = \zeta$. The unsteady Bernoulli equation (62) for a constant force of gravity $\Phi = gz$ is

$$p/\rho + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{\partial \phi}{\partial t} + gz = C(t),$$

and so the third boundary condition becomes

$$p_{atm}/\rho + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{\partial \phi}{\partial t} + g\zeta = C(t), \qquad \text{on } z = \zeta. \tag{113}$$

This concludes our description of the fully nonlinear problem.

5.2 Linear gravity waves

By the small amplitude assumption, $|\zeta| \ll h$ and $\left|\dfrac{\partial \zeta}{\partial x}\right| \ll 1$. If we assume that the flow is driven by the motion of the free surface (no additional driving from below the surface), it must be that the flow is also week: $|\phi| \ll 1$. Thus we will throw away all terms *quadratic* in the two variables ζ and ϕ, i.e. containing ζ^2, ϕ^2, or products $\zeta\phi$. Note that (113) contains the velocity at quadratic (nonlinear) order, which is a remnant of the nonlinear character of the Euler equation, as it remains in Bernoulli's equation. The kinematic boundary condition (112) contains $\dfrac{\partial \phi}{\partial x}\dfrac{\partial \zeta}{\partial x}$, which is quadratic as well. Both terms will be neglected in our linearised treatment.

But this represents only one type of nonlinearity. The most significant problem is that in the formulation of the boundary conditions, the position of the free surface is part of the solution, so don't know where to apply the boundary conditions ! We can deal with this difficulty by linearising about $z = 0$ using Taylor's expansion about $z = 0$ for all z-dependent terms, for example

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=\zeta} = \left.\frac{\partial \phi}{\partial t}\right|_{z=0} + \zeta \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)_{z=0} + \dots .$$

The second term on the right is quadratic, and will be neglected. Thus we observe that linearizing the boundary conditions implies that instead of at $z = \zeta(x,t)$, the boundary condition is imposed at the equilibrium position $z = 0$. Thus we obtain

(i) On $z = -h$, $\dfrac{\partial \phi}{\partial z} = 0$

(ii) On $z = 0$, $\dfrac{\partial \zeta}{\partial t} = \dfrac{\partial \phi}{\partial z}$

(iii) On $z = 0$, $\dfrac{\partial \phi}{\partial t} + g\zeta = C(t) - p_{atm}/\rho$.

The last condition can be prettyfied by redefining the potential according to

$$\phi = \phi' - p_{atm}t/\rho + \int^t C(t')dt'.$$

The extra terms are time dependent only, so they do not affect the flow velocities, Laplace's equation or the boundary conditions. So now (dropping primes) (iii) is replaced with

(iii) On $z = 0$, $\dfrac{\partial \phi}{\partial t} + g\zeta = 0$.

The linearised pressure *in* the fluid is

$$\frac{p(x,z,t)}{\rho} + \frac{\partial \phi}{\partial t} + gz = C(t)$$

and because of $\phi \to \phi'$ this transforms to

$$\frac{p(x,z,t) - p_{atm}}{\rho} = -\frac{\partial \phi}{\partial t} - gz. \tag{114}$$

Now we look for particular solutions to $\boldsymbol{\nabla}^2\phi = 0$ with (i), (ii), (iii) motivated by observations, which describe a propagating wave.

Such a propagating wave of amplitude H is described by a solution of the form

$$\zeta(x,t) = H\cos(kx - \omega t), \tag{115}$$

where an arbitrary phase factor δ could have been added as well. The solution (115) is periodic both in time (with period $\tau = 2\pi/\omega$) and space (with wavelength $\lambda = 2\pi/k$). Most importantly, it moves to the *right* with speed $c = \omega/k$, where c is called the phase speed. To see that, let $\xi = x - ct$, so that

$$\zeta = H\cos(k\xi)$$

is unchanged for ξ constant. But this means that

$$0 = \frac{d\xi}{dt} = \frac{dx}{dt} - c,$$

so c is the speed at which the point x is moving.

Now we need the potential ϕ, which describes the velocity field. Since $\dfrac{\partial \phi}{\partial t} = -g\zeta$ reasonable to write

$$\phi(x,z,t) = \sin(kx - \omega t)Z(z)$$

for some $Z(z)$ – this is a separation solution $\phi = X(x)Z(z)$. Then from $\triangle\phi = 0$,

$$-k^2 \sin(kx - \omega t)Z(z) + \sin(kx - \omega t)Z''(z) = 0$$

and so $Z''(z) - k^2 Z(z) = 0$. Then

$$Z(z) = A \cosh k(z + h) + B \sinh k(z + h),$$

for constants A, B. Because of (i), need $B = 0$, so

$$\phi(x, z, t) = A \sin(kx - \omega t) \cosh k(z + h). \tag{116}$$

We still have (ii) and (iii) to apply, but only A to find. The second condition will lead to an equation for ω. First from (iii), inserting (115) and (116) into $g\zeta = -\partial\phi/\partial t|_{z=0}$, we find

$$gH \cos(kx - \omega t) = A\omega \cos(kx - \omega t) \cosh kh,$$

so that

$$A = gH/(\omega \cosh kh). \tag{117}$$

In other words, the potential is

$$\phi(x, z, t) = \frac{gH}{\omega \cosh kh} \sin(kx - \omega t) \cosh k(z + h). \tag{118}$$

Then from (ii), inserting (115) and (118) into $\partial\phi/\partial z|_{z=0} = \partial\zeta/\partial t$ results in

$$kA \sin(kx - \omega t) \sinh kh = \omega H \sin(kx - \omega t).$$

This implies

$$\omega H = k \left(\frac{gH}{\omega \cosh kh} \right) \sinh kh,$$

or

$$\frac{\omega^2 h}{g} = kh \tanh kh, \tag{119}$$

which is called the *dispersion relation* of the wave, plotted in Fig. 58.

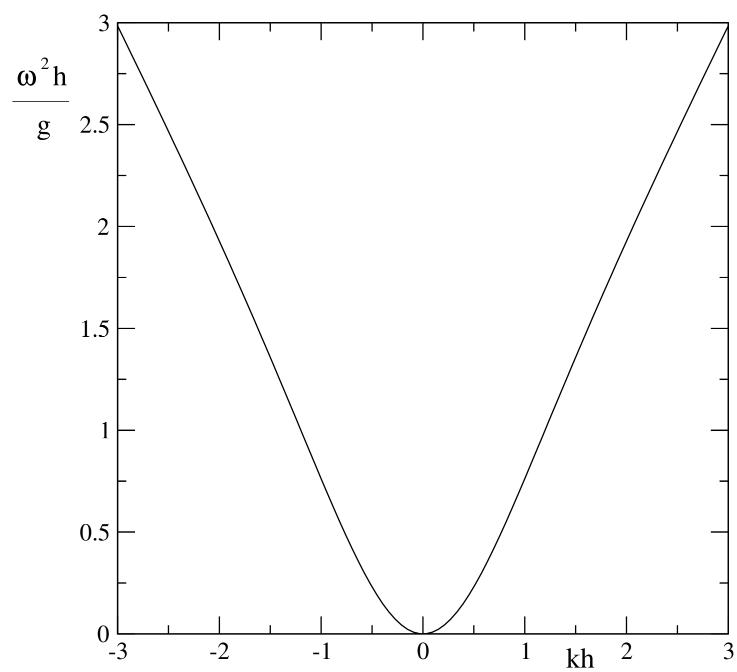


Figure 58: The dispersion relation (119).

The relation (119) is the principal result of our calculation. It establishes how the frequency of a <u>monochromatic</u> wave is related to its wavelength. Very roughly, large $\lambda \Rightarrow$ small $k \Rightarrow$ small $\omega \Rightarrow$ large $\tau = 2\pi/\omega$. So long wavelengths imply long periods, and vice versa.

If the dispersion relation between ω and k is non-linear, one speaks of a dispersive problem: parts of a wave containing different frequencies travel at different speeds, and thus *disperse*. If the relation is linear (as for exlectromagnetic waves), the phase speed is a constant.

We remark that changing $k \to -k$, (119) still holds. Then (115) becomes

$$\zeta(x,t) = H\cos(kx + \omega t),$$

which is a wave travelling to the left, with speed $c = -\omega/k$. The constant $k = 2\pi/\lambda$ is called the wavenumber (the number of wavelengths that can be fit into 2π).

5.2.1 Two special cases

Two important special cases of (119) exist:

1. **Long wavelengths** This is the case $kh \ll 1$ (or $\lambda/h \gg 1$), which means that the wave length is long compared to the depth. Then $\tanh kh \approx kh$ or

   $$\omega = \sqrt{gh}k,$$

   so the dispersion relation is a linear function. The constant of proportionality is the (constant) wave speed $c = \omega/k = \sqrt{gh}$. Evidently, the shallow water problem has *no dispersion.*

2. **Short wavelengths** If on the other hand $kh \gg 1$ (or $\lambda/h \ll 1$), the water is deep relative to the length of the wave. Then $\tanh kh \approx 1$ and the dispersion relation is

   $$\omega = \sqrt{gk}.$$

   Thus the wave speed $c = \omega/k = \sqrt{g/k} = \sqrt{g\lambda/2\pi}$ does depend on the wavelength, and there is dispersion. Since $\cosh k(z+h) \approx e^{kz}e^{kh}/2$ the potential (118) becomes

   $$\phi(x,z,t) = \frac{H}{c}e^{kz}\sin(kx - \omega t). \tag{120}$$

   The amplitude decays exponentially as one goes away from the surface.

5.3 Particle trajectories

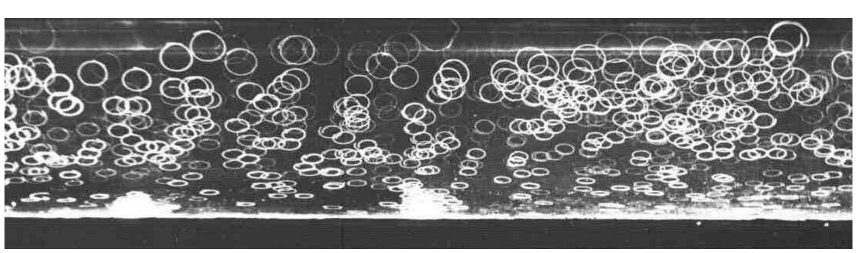


Figure 59: The trajectories of particles underneath a wave.

The fact that a traveling wave moves at a speed $c = \omega/k$ does not mean that particles inside the fluid also travel at this speed! Since the amplitude is small, we consider particles at a position $(x + \xi, z + \eta)$, where ξ and η are small departures from the mean position (x, z) of the particle. Using (116), the flow velocities are

$$u = kA \cos k(x+\xi-ct) \cosh k(z+\eta+h), \quad w = kA \sin k(x+\xi-ct) \sinh k(z+\eta+h). \quad (121)$$

Now for small ξ and η, we can put $\xi = \eta = 0$ in (121) to obtain the equations of motion

$$\frac{d\xi}{dt} = kA \cos k(x - ct) \cosh k(z + h), \quad \frac{d\eta}{dt} = kA \sin k(x - ct) \sinh k(z + h),$$

which we can integrate to yield the trajectory

$$\xi = -\frac{A}{c} \sin k(x - ct) \cosh k(z + h), \quad \eta = \frac{A}{c} \cos k(x - ct) \sinh k(z + h). \quad (122)$$

Thus we find that as long as the amplitude A of the wave is small, the motion (122) of the particles is small as well. Hence we were justified in putting $\xi = \eta = 0$ as a first approximation. In Section 5.4 below we will pursue the expansion to the next (quadratic) order.

Taking the square of each of the components of (122), we find

$$\frac{\xi^2}{\cosh^2 k(z + h)} + \frac{\eta^2}{\sinh^2 k(z + h)} = \frac{A^2}{c^2}, \quad (123)$$

which is the equation for an ellipse with major and minor semi-axes (in the x and z directions)

$$a = \frac{A}{c} \cosh k(z + h), \quad a = \frac{A}{c} \sinh k(z + h).$$

Using the expression (117) for A and the dispersion relation (119), $A/c = H/\sinh kh$, and so

$$a = H\frac{\cosh k(z + h)}{\sinh kh}, \quad b = H\frac{\sinh k(z + h)}{\sinh kh}, \quad (124)$$

while the ratio between the two axes is

$$\frac{b}{a} = \tanh k(z + h). \quad (125)$$

This means that if the point of observation is far from the bottom ($k(z + h) \gg 1$), $a \approx b$ and the trajectory is a circle. This is clearly visible in Fig. 59, where particle trajectories are visible as bright lines. The trajectories near the free surface are close to being circles, while closer to the lower boundary (where $z+h$ becomes small), trajectories are flattened in the vertical direction. The size of the ellipse decreases as well.

5.4 Stokes drift

A major conclusion of the previous section is that although the wave is propagating, there is *no* particle transport, as particles move on closed orbits. G.G. Stokes discovered that there is in fact a little bit of transport in the wave direction, since particle paths are not *exactly* closed. To discover this, one has to go to next order in a perturbation expansion in ξ, η.

To simplify the discussion, we consider the case (120) of *infinite* depth, for which the equation of motion reads

$$\frac{d\xi}{dt} = \overline{A}c e^{k(z+\eta)} \cos k(x+\xi-ct), \quad \frac{d\eta}{dt} = \overline{A}c e^{k(z+\eta)} \sin k(x+\xi-ct), \tag{126}$$

where $\overline{A} = \frac{gHk}{c\omega} = \frac{gHk^2}{\omega^2} = Hk$ is a dimensionless number. Expanding for small values of ξ,η, one finds

$$\frac{d\xi}{dt} = \overline{A}c\left[e^{kz}\cos k(x-ct) + \eta k e^{kz}\cos k(x-ct) - \xi k e^{kz}\sin k(x-ct)\right],$$

and

$$\frac{d\eta}{dt} = \overline{A}c\left[e^{kz}\sin k(x-ct) + \eta k e^{kz}\sin k(x-ct) + \xi k e^{kz}\cos k(x-ct)\right].$$

Inserting the leading order expressions

$$\xi = -\frac{\overline{A}}{k}e^{kz}\sin k(x-ct), \quad \eta = \frac{\overline{A}}{k}e^{kz}\cos k(x-ct),$$

this results in

$$\frac{d\xi}{dt} = \overline{A}c e^{kz}\cos k(x-ct) + \overline{A}^2 c e^{2kz}, \quad \frac{d\eta}{dt} = \overline{A}c e^{kz}\sin k(x-ct) + O(\overline{A}^3).$$

This is of course solved to give

$$\xi = -\frac{\overline{A}}{k}e^{kz}\sin k(x-ct) + \overline{A}^2 c e^{2kz} t, \quad \eta = \overline{A}e^{kz}\cos k(x-ct). \tag{127}$$

To leading order, the trajectories are circles, and no global transport takes place. The only, but crucial difference is that at quadratic order, particles are pushed forward an extra horizontal distance, so the circles do not close. On average, particles are pushed along with the wave with a drift velocity of

$$U_S = \overline{A}^2 c e^{2kz}, \tag{128}$$

which is known as Stokes drift.

5.5 Oscillations in a container

Liquids readily slosh back and forth in a closed container (e.g. tea in a tea-cup). There are many associated important practical problems: sloshing in road tankers, water on decks of ships, resonance in harbours, etc.

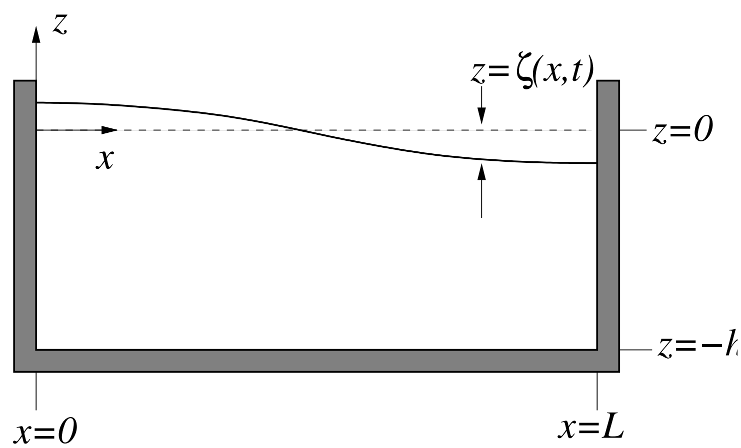


Figure 60: Standing waves in a container

As a model for such situations, consider a two-dimensional rectangular box with rigid walls at $x = 0, L$ and a bottom on $z = -h$, filled with fluid to $z = 0$, see Fig. 60. We use the small-amplitude theory from before, so that linearised equations are $\boldsymbol{\nabla}^2\phi = 0$ for (x, y, z) in box,

(i) $\dfrac{\partial \phi}{\partial z} = 0$ on $z = -h$.

(ii) Kinematic: $\dfrac{\partial \zeta}{\partial t} = \dfrac{\partial \phi}{\partial z}$ on $z = 0$.

(iii) Dynamic: $\dfrac{\partial \phi}{\partial t} = -g\zeta$ on $z = 0$.

Also need no-flow conditions (kinematic) on walls:

(iv) $\dfrac{\partial \phi}{\partial x} = 0$ on $x = 0, L$;

However now we no longer have travelling waves, but are interested in describing a phenomenon that is periodic in time, so we write in two dimensions:

$$\zeta(x, t) = \zeta(x) \sin \omega t$$

and assume we can separate variables in ϕ:

$$\phi(x, z, t) = X(x)Z(z) \cos \omega t.$$

Then Laplace's equation gives

$$X''(x)Z(z) \cos \omega t + Z''(z)X(x) \cos \omega t = 0$$

which implies

$$X''(x)Z(z) + Z''(z)X(x) = 0$$

This separates:

$$\frac{X''(x)}{X(x)} = -\frac{Z''(z)}{Z(z)} = -k^2$$

where $-k^2$ is the separation constant.

Solving for $Z(z)$, with (i) gives $Z(z) = A\cosh k(z+h)$ as before. Combining (ii) and (iii) to eliminate ζ (that's g(ii) $+\partial/\partial t$(iii)) gives

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\phi}{\partial z}, \qquad \text{on } z=0$$

which implies

$$-\omega^2 X(x)Z(0)\cos\omega t = -gX(x)Z'(0)\cos\omega t$$

and it follows that $\omega^2/g = k\tanh kh$ as before. (We expect this, as the vertical structure of the fluid is independent of the vertical lateral walls)

Now solving for $X(x)$ gives

$$X(x) = B\cos(kx) + C\sin(kx)$$

subject to (from (iv)), $X'(0) = X'(L) = 0$. Easy to show that must have $C = 0$ and $k = n\pi/L$ and so

$$X(x) = B\cos(n\pi x/L)$$

So pulling everything together we have

$$\phi(x,z,t) = A'\cos(n\pi x/L)\cosh k(z+h)\cos\omega t$$

for some constant $A' = AB$ while from (iii) the free surface is given by

$$\zeta(x,t) = \frac{\omega A'\cosh kh}{g}\cos(n\pi x/L)\sin\omega t$$

and here, $k = n\pi/L$, so that (119) reads

$$\omega^2/g = (n\pi/L)\tanh(n\pi h/L) \equiv \omega_n^2/g, \qquad n = 0,1,2,\ldots$$

and therefore defines a set of discrete wave frequencies at which these oscillations may occur. Here, n is a mode number and tells you how many oscillations are occurring across the box. Crucially, by considering waves in a finite box, we have selceted a *discrete spectrum* of allowed frequencies.

We cannot have $n = 0$ as then $\omega = 0$ and $\phi = 1$, and $\zeta = 0$. So there is no motion in the fluid. (In fact, you can discount a sloshing mode with no x-dependence – a flat surface oscillating up and down – as it would violate mass conservation in the tank). The fundamental and higher order modes are illustrated in Fig. 60. The higher n, the higher the frequency. As $n \to \infty$, $\tanh(n\pi h/L) \to 1$ and so $\omega_n \to \sqrt{gn\pi/L} = \sqrt{gk}$. For large n, the discreteness of the spectrum hardly plays a role any more: any wavelength is approximated closely by one of the discrete modes.

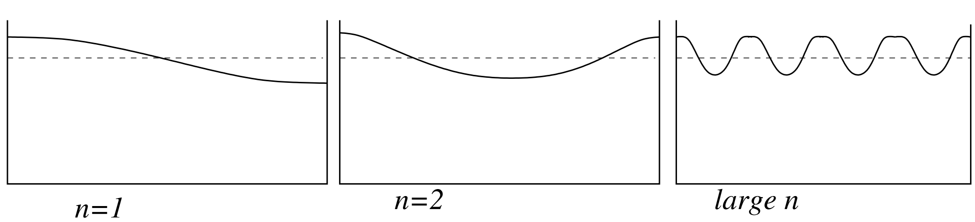


Figure 61: Different modes in a container

The Fundamental frequency is the lowest frequency, given here by $n = 1$.

T_1 [s]	h [cm]	$\frac{\omega_1^2 L}{g}$	$\pi \tanh \frac{\pi h}{L}$
4.15	0.7	0.17	0.093
2.7	1.4	0.41	0.19
2.1	2.3	0.68	0.31

The Table gives experimental data for measurements in a rectangular tank (see picture), whose length was $L = 74$ cm. One end of the tank was lifted, to excite the fundamental sloshing mode. The third and fourth columns compare experiment and theory, as

$$\frac{\omega_1^2 L}{g} = \pi \tanh \frac{\pi h}{L},$$

and $T_1 = 2\pi/\omega_1$. Remarkably, the period *decreases* with increasing depth h. The trend is reproduced correctly, but there is a consistent discrepancy by a factor of two, perhaps due to the fact that the tank is too long, so we didn't really excite the findamental mode...

Appendix A: Vector calculus

We shall revise some vectors operations that you should have already met before this course. We will be using suffix notation throughout this course

A.1 Suffix notation and summation convention

Suppose that we have two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then the dot product is defined to be

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i \qquad \text{or, more simply, write} \qquad \mathbf{u} \cdot \mathbf{v} = u_i v_i$$

(drop the summation symbol on the understanding that repeated suffices imply summation.

Defn A.1.2: The Kronecker delta is defined by

$$\delta_{ij} = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right\}$$

So in summation convention, $\delta_{ij} a_j = a_i$ since

$$\delta_{ij} a_j \equiv \sum_{j=1}^{3} \delta_{ij} a_j = a_i$$

Examples:

1. $\delta_{ii} = 3$
2. $\delta_{ij} u_i v_j = u_j v_j \equiv \mathbf{u} \cdot \mathbf{v}$.

Defn A.1.3: The antisymmetric symbol ϵ_{ijk} is defined by

- $\epsilon_{123} = 1$
- ϵ_{ijk} is zero if there are any repeated suffices. E.g. $\epsilon_{113} = 0$.
- Interchanging any two suffices reverses the sign: e.g. $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji}$
- Above implies invariant under cyclic rotation of suffices: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$

With this definition all 27 permutations are defined. There are only 6 non-zero components,

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \qquad \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1,$$

Defn A.1.4: The cross product is defined by

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{r}} + (u_3 v_1 - u_1 v_3)\hat{\mathbf{y}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{z}}$$

But can now be written in component form as

$$w_i = \epsilon_{ijk} u_j v_k$$

where summation over j and k occurs. [Check].

Example: Consider the triple product,

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = w_i \epsilon_{ijk} u_j v_k$$

where summation is over i, j, k so result is scalar. It follows that

$$\begin{aligned}
\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= \epsilon_{jki} w_i u_j v_k = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \\
&= \epsilon_{kij} w_i u_j v_k = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) \\
&= -\epsilon_{jik} w_i u_j v_k = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \\
&= -\epsilon_{ikj} w_i u_j v_k = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})
\end{aligned}$$

Defn A.1.5: The double product of ϵ_{ijk} is

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

[Check]

Defn A.1.6: The vector triple product is defined by the result

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \cdot \mathbf{v}) \mathbf{u} - (\mathbf{w} \cdot \mathbf{u}) \mathbf{v}$$

Proof:

$$\begin{aligned}
[\mathbf{w} \times (\mathbf{u} \times \mathbf{v})]_i &= \epsilon_{ijk} w_j [\mathbf{u} \times \mathbf{v}]_k \\
&= \epsilon_{ijk} w_j \epsilon_{klm} u_l v_m \\
&= \epsilon_{kij} \epsilon_{klm} w_j u_l v_m \\
&= (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) w_j u_l v_m \\
&= w_j u_i v_j - w_j u_j v_i = (\mathbf{w} \cdot \mathbf{v}) u_i - (\mathbf{w} \cdot \mathbf{u}) v_i
\end{aligned}$$

True for $i = 1, 2, 3$, hence result.

A.2 Differential operations

Here we consider operations on a function $\phi(\mathbf{r})$ and a vector field $\mathbf{f}(\mathbf{r})$ where $\mathbf{r} = (x_1, x_2, x_3)$. One can regard $\boldsymbol{\nabla}$ as the vector operator $\left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3} \right)$. Without using any information (but always remembering the true meaning of the symbol!), one can also use the much sleaker notation $\boldsymbol{\nabla} \equiv (\partial_1, \partial_2, \partial_3)$. I will usually do that whenever using summation convention. Then

- The gradient is $\boldsymbol{\nabla} \phi$. So $[\boldsymbol{\nabla} \phi]_i = \dfrac{\partial \phi}{\partial x_i} \equiv \partial_i \phi$
- The divergence is $\boldsymbol{\nabla} \cdot \mathbf{f} = \dfrac{\partial f_i}{\partial x_i} \equiv \partial_i f_i$ (in summation convention)

- The <u>curl</u> is $\boldsymbol{\nabla} \times \mathbf{f}$ where $[\boldsymbol{\nabla} \times \mathbf{f}]_i = \epsilon_{ijk} \frac{\partial f_k}{\partial x_j} \equiv \epsilon_{ijk} \partial_j f_k$.

Examples:

1. $[\boldsymbol{\nabla}(x_j)]_i = \partial_j x_i = \delta_{ij}$
2. $\boldsymbol{\nabla} \cdot \mathbf{r} = \partial_i x_i = \delta_{ii} = 3$
3. $[\boldsymbol{\nabla} \times \mathbf{r}]_i = \epsilon_{ijk} \partial_k x_j = \epsilon_{ijk} \delta_{kj} = \epsilon_{ijj} = 0$
4. $\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r}$, where $r^2 \equiv \mathbf{r} \cdot \mathbf{r}$.

   Proof: $[\boldsymbol{\nabla} r]_i = \partial_i \sqrt{x_j x_j} = \dfrac{\partial_i x_j^2}{2\sqrt{x_j^2}} = \dfrac{x_j \partial_i x_j}{r} = \dfrac{x_j \delta_{ij}}{r} = \dfrac{x_i}{r}$.

A.2.1 Useful vector identities

1. $\boldsymbol{\nabla} \cdot (\phi \mathbf{f}) = \partial_i(\phi f_i) = \phi \partial_i f_i + f_i \partial_i \phi = \phi \boldsymbol{\nabla} \cdot \mathbf{f} + \mathbf{f} \cdot \boldsymbol{\nabla} \phi$
2. $\boldsymbol{\nabla} \times (\phi \mathbf{f}) = \phi(\boldsymbol{\nabla} \times \mathbf{f}) + (\boldsymbol{\nabla} \phi \times \mathbf{f})$

   Proof: $[\boldsymbol{\nabla} \times (\phi \mathbf{f})]_i = \epsilon_{ijk} \partial_j(\phi f_k) = \phi \epsilon_{ijk} \partial_j f_k + f_k \epsilon_{ijk} \partial_j \phi = \phi[\boldsymbol{\nabla} \times \mathbf{f}]_i + [\boldsymbol{\nabla} \phi \times \mathbf{f}]_i$.
3. $\mathbf{f} \times (\boldsymbol{\nabla} \times \mathbf{f}) = \boldsymbol{\nabla}(\frac{1}{2}\mathbf{f} \cdot \mathbf{f}) - (\mathbf{f} \cdot \boldsymbol{\nabla})\mathbf{f}$

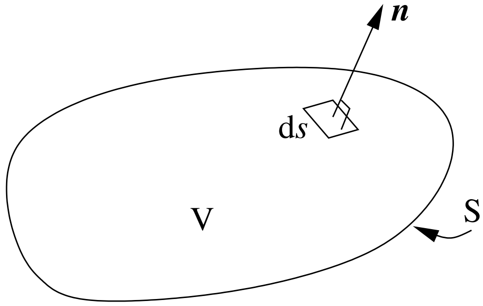
   Proof:

   $$\begin{aligned} [\mathbf{f} \times (\boldsymbol{\nabla} \times \mathbf{f})]_i &= \epsilon_{ijk} f_j [\boldsymbol{\nabla} \times \mathbf{f}]_k = \epsilon_{ijk} f_j \epsilon_{klm} \partial_l f_m = \epsilon_{kij} \epsilon_{klm} f_j \partial_l f_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) f_j \partial_l f_m = f_j \partial_i f_j - f_j \partial_j f_i = \partial_i \frac{1}{2} f_j f_j - (\mathbf{f} \cdot \boldsymbol{\nabla}) f_i \end{aligned}$$

   Hence result.

A.3 Integral results

A.3.1 The divergence theorem



Very important. Consider a volume V bounded by a closed surface S with outward unit normal $\mathbf{n}$. Then for a vector field $\mathbf{f}(\mathbf{x})$

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{f} dV = \int_S \mathbf{f} \cdot \mathbf{n} dS$$

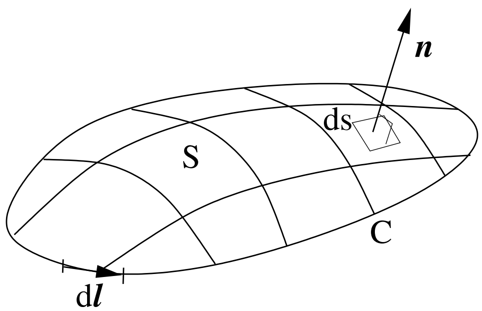
Corollary: Let $\mathbf{f}(\mathbf{r}) = \mathbf{a}\phi(\mathbf{r})$ where $\mathbf{a}$ is an arbitrary constant vector, and $\phi(\mathbf{r})$ a scalar function. Since $\boldsymbol{\nabla} \cdot (\mathbf{a}\phi(\mathbf{r})) = \mathbf{a} \cdot \boldsymbol{\nabla}\phi(\mathbf{r})$, so the divergence theorem reduces to

$$\mathbf{a} \cdot \int_V \boldsymbol{\nabla}\phi dV = \mathbf{a} \cdot \int_S \phi \mathbf{n} dS$$

True for any $\mathbf{a}$, so

$$\int_V \boldsymbol{\nabla}\phi dV = \int_S \phi \mathbf{n} dS$$

A.3.2 Stokes' theorem



Let C be a closed curve bounding a surface S with unit normal $\mathbf{n}$.
Then for a vector field $\mathbf{f}(\mathbf{r})$,

$$\oint_C \mathbf{f} \cdot d\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{f}) \cdot \mathbf{n} dS$$

where $d\mathbf{l}$ is a line element on C.

Corollary: Let $\mathbf{f}(\mathbf{r}) = \mathbf{a}\phi(\mathbf{r})$ where $\mathbf{a}$ is an arbitrary constant vector. Then

$$\oint_C \mathbf{a}\phi \cdot d\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{a}\phi) \cdot \mathbf{n} dS$$

and from §A.2.1(2), $(\boldsymbol{\nabla} \times \mathbf{a}\phi) = \phi(\boldsymbol{\nabla} \times \mathbf{a}) + (\boldsymbol{\nabla}\phi \times \mathbf{a}) = (\boldsymbol{\nabla}\phi \times \mathbf{a})$. Using the vector triple product result, $(\boldsymbol{\nabla}\phi \times \mathbf{a}) \cdot \mathbf{n} = -\mathbf{a} \cdot (\boldsymbol{\nabla}\phi \times \mathbf{n})$ and then

$$\mathbf{a} \cdot \oint_C \phi d\mathbf{l} = -\mathbf{a} \cdot \int_S \boldsymbol{\nabla}\phi \times \mathbf{n} dS$$

Therefore

$$\oint_C \phi d\mathbf{l} = -\int_S \boldsymbol{\nabla}\phi \times \mathbf{n} dS$$

Appendix B: Curvilinear coordinate systems

Many problems can be approached more simply by choosing a coordinate system that fits a given geometry. Instead of writing the position vector $\mathbf{r}$ as a function of Cartesian coordinates (x, y, z), $\mathbf{r}$ is now written as function of three new coordinates: $\mathbf{r}(q_1, q_2, q_3)$. The coordinate lines are swept out by varying one of the coordinates, keeping the other two constant. We will deal only with the by far most important case of *orthogonal* coordinate systems, in which the coordinate lines always intersect one another at right angles.

Evidently, $\dfrac{\partial \mathbf{r}}{\partial q_i}$ is a vector which points in the direction of the i-th coordinate line, see the figure. If each of these vectors are normalized to unity, we obtain the local basis system:

$$\hat{\mathbf{q}}_i = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial q_i}, \quad h_i \equiv \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|. \tag{129}$$

The quantities $h_i(q_1, q_2, q_3)$ are called *scale factors* or *metric coefficients*. The fact that the coordinate system is orthogonal means that all $\hat{\mathbf{q}}_i$, computed at a point (q_1, q_2, q_3), are orthogonal. However, the direction of $\hat{\mathbf{q}}_i$ of course changes as one goes along the coordinate lines.

Example: The cylindrical polar coordinate system.
The position vector is

$$\mathbf{r} = (r\cos\theta, r\sin\theta, z)\,,$$

and the coordinates are (r, θ, z). Thus the scale factors become $h_1 = 1$, $h_2 = r$, and $h_3 = 1$, and the local basis is

$$\begin{aligned}
\hat{\mathbf{r}} &= \frac{\partial \mathbf{r}}{\partial r} = (\cos\theta, \sin\theta, 0)\,, \\
\hat{\boldsymbol{\theta}} &= \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} = (-\sin\theta, \cos\theta, 0)\,, \\
\hat{\mathbf{z}} &= \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).
\end{aligned}$$

It is clear that $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ are indeed mutially orthogonal.

Remark: As the coordinates are varied by δq_1, δq_2, and δq_3, respectively, $\mathbf{r}$ describes a volume whose sides are orthogonal. The length of each side is $h_i \delta q_i$, and thus the volume of the cuboid is $d^3x = h_1 h_2 h_3 dq_1 dq_2 dq_3$. Thus *integration* in a curvilinear coordinate system is achieved by the formula

$$\int_V f(\mathbf{r}) d^3x = \int_V f(q_1, q_2, q_3) h_1 h_2 h_3 dq_1 dq_2 dq_3. \tag{130}$$

In cylindrical polars, the volume element is $r dr d\theta dz$.

To do vector calculus in the curvilinear coordinate system, we have to work out what the $\boldsymbol{\nabla}$-operator is. For any scalar function $f(\mathbf{r})$ (and suspending the summation convention for the moment), we have

$$\frac{\partial f}{\partial q_i} = \sum_{j=1}^{3} \frac{\partial x_j}{\partial q_i}\frac{\partial f}{\partial x_j} = \left(\frac{\partial \mathbf{r}}{\partial q_i} \cdot \boldsymbol{\nabla}\right) f = h_i \left(\hat{\mathbf{q}}_i \cdot \boldsymbol{\nabla}\right) f$$

by the chain rule, and using (129). But up to the factor h_i, the object on the right-hand side is the q_i component of $\boldsymbol{\nabla} f$. This means that in the local basis, $\boldsymbol{\nabla}$ has the representation

$$\boldsymbol{\nabla} = \frac{\hat{\mathbf{q}}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{\mathbf{q}}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{\mathbf{q}}_3}{h_3}\frac{\partial}{\partial q_3}. \tag{131}$$

For example in cylindrical polars, the gradient of a scalar function ϕ is

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{z}},$$

in agreement with chapter 0.

Now we are almost there. The only but crucial thing that remains to be considered is the fact that both scale factors and unit vectors depend on the coordinates q_i, so they need to be differentiated. In particular, we want to represent $\dfrac{\partial \hat{\mathbf{q}}_i}{\partial q_j}$ in terms of the basis vectors. It is not difficult to write down general expressions for the derivatives of the basis vectors if only the h_i are known, see exercises. However, for a given coordinate system, it is generally much easier to simply calculate the derivatives directly. For example, in cylindrical polars

$$\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\hat{\mathbf{r}},$$

and all the other derivatives are zero.

Now it is easy to derive the other formulae in chapter 0. In cylindrical polars, the Laplacian becomes

$$\triangle\phi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left(\hat{\mathbf{r}}\frac{\partial \phi}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial \phi}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial \phi}{\partial z}\right) =$$

$$(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\left(\hat{\boldsymbol{\theta}}\cdot\frac{\partial \hat{\mathbf{r}}}{\partial \theta}\right)\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\left(\hat{\boldsymbol{\theta}}\cdot\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta}\right)\frac{\partial \phi}{\partial \theta} + \left(\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}\right)\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{z}})\frac{\partial^2\phi}{\partial z^2} =$$

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} + \frac{\partial^2\phi}{\partial z^2}.$$

In local coordinates, a vector field is written as $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$. Then the divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\left(u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}\right) = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

again in agreement with chpater 0.

Appendix C: The streamfunction

If $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, then it follows that there exists a vector field $\mathbf{A}(\mathbf{r}, t)$ s.t.

$$\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A}.$$

Conversely, it is clear from this representation that the flow is incompressible. The representation is particularly useful in two dimensions, in which case $\mathbf{A}$ can be written in terms of a single scalar function, the streamfunction. By definition, the streamfunction is constant along streamlines.

(i) Cartesians: Consider Cartesian coordinates (x, y, z), and take the flow in the (x, y)-plane. Then $\mathbf{A} = \psi(x, y, t)\hat{\mathbf{z}}$, and

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

Now confirm that ψ is indeed constant along streamlines, which are defined by

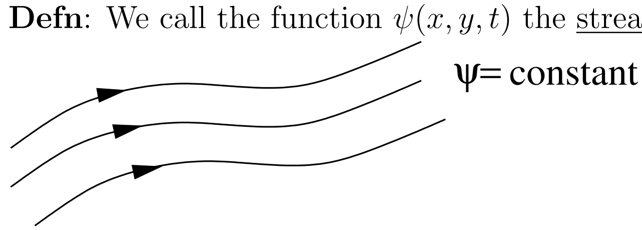
$$\frac{d\mathbf{r}}{ds} = \lambda \mathbf{u},$$

and thus $dx = \lambda u ds$ and $dy = \lambda v ds$. It follows that

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y = v\mathrm{d}x - u\mathrm{d}y = 0,$$

if indeed ψ is varied along streamlines. Thus ψ is constant along a streamlines, as advertised.

Defn: We call the function $\psi(x, y, t)$ the streamfunction of the flow.



(ii) Cylindrical polar coordinates: Now consider flow in the same two dimensional plane $z = 0$, which is represented in cylindrical polars (r, θ, z). The relation to Cartesian coordinates is $x = r\cos\theta, y = r\sin\theta$. Since the stream function is defined completely by the geometry of the streamlines, it must be the same in any coordinate system. In other words if $\psi_c(x, y, t)$ is the Cartesian version, and $\psi_p(x, y, t)$ the streamfunction in polar coordinates, then $\psi_p(r, \theta, t) = \psi_c(r\cos\theta, r\sin\theta, t)$.

As before, the vector potential is $\mathbf{A} = \psi(r, \theta, t)\hat{\mathbf{z}}$, and thus from the curl in polar coordinates

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r},$$

where $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}}$.

Now verify that $\psi(r, \theta)$ is indeed constant along streamlines. In polar coordinates, $\mathbf{r} = r\hat{\mathbf{r}}$, and thus

$$\frac{d\mathbf{r}}{ds} = \frac{dr}{ds}\hat{\mathbf{r}} + r\frac{d\theta}{ds}\frac{d\hat{\mathbf{r}}}{d\theta} = \frac{dr}{ds}\hat{\mathbf{r}} + r\frac{d\theta}{ds}\hat{\boldsymbol{\theta}},$$

where we have used that $\dfrac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}}$. In other words, we find that $dr = \lambda u_r ds$ and $rd\theta = \lambda u_\theta ds$. Now as before,

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial r}\mathrm{d}r + \frac{\partial \psi}{\partial \theta}\mathrm{d}\theta = -u_\theta \mathrm{d}r + r u_r \mathrm{d}\theta = 0,$$

if ψ is indeed varied along streamlines.

Appendix D Some simple flows and their potentials

D.1 Two dimensional flows

We will give the answer either in Cartesians or polars, depending on which is more convenient.

Reminder:

(i) in Cartesians, $\mathbf{u} = (u, v, 0)$ and $u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$, $v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$ where ϕ is the velocity potential and ψ is the streamfunction.

(ii) In polars, $\mathbf{u} = (u_r, u_\theta, 0)$ and $u_r = \dfrac{\partial \phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$, $u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = -\dfrac{\partial \psi}{\partial r}$.

Type of flow	flow field $\mathbf{u}$	potential ϕ	streamfunction ψ	complex pot. w
Uniform stream parallel to x axis	$U\hat{\mathbf{r}}$	Ux	Uy	Uz
Uniform stream at angle α to x axis	$U\cos\alpha\hat{\mathbf{r}} +$ $U\sin\alpha\hat{\mathbf{y}}$	$Ur\cos(\theta-\alpha)$	$Ur\sin(\theta-\alpha)$	$Uze^{-i\alpha}$
Stagnation point flow at origin	$u = kx$ $v = -ky$	$\dfrac{k}{2}(x^2-y^2)$	kxy	$\dfrac{k}{2}z^2$
Line source, strength m, at $r=0$	$\dfrac{m\hat{\mathbf{r}}}{2\pi r}$	$\dfrac{m}{2\pi}\log r$	$\dfrac{m\theta}{2\pi}$	$\dfrac{m}{2\pi}\ln z$
Line vortex, circulation Γ, at $r=0$	$\dfrac{\Gamma\hat{\boldsymbol{\theta}}}{2\pi r}$	$\dfrac{\Gamma\theta}{2\pi}$	$-\dfrac{\Gamma}{2\pi}\log r$	$-\dfrac{i\Gamma}{2\pi}\ln z$
Horizontal dipole, strength μ, at $r=0$	$-\dfrac{\mu}{2\pi r^2}\left(\hat{\mathbf{z}} - 2\dfrac{z}{r}\hat{\mathbf{r}}\right)$	$-\mu\dfrac{\cos\theta}{2\pi r}$	$\mu\dfrac{\sin\theta}{2\pi r}$	$-\dfrac{\mu}{2\pi z}$

D.2 Axisymmetric flows

We use cylindrical or spherical polars, whichever is more convenient. In cylindrical coordinates, (r, θ, z), $\mathbf{u} = (u_r, u_\theta, u_z)$; in terms of the potential $\phi(r, z)$:

$$u_r = \frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad u_z = \frac{\partial \phi}{\partial z} = \frac{1}{r}\frac{\partial \phi}{\partial r}.$$

In spherical polar coordinates, (r, θ, φ), $\mathbf{u} = (u_r, u_\theta, u_\varphi)$; in terms of the potential $\phi(r, \theta)$:

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r^2\sin\theta}\frac{\partial \Psi}{\partial \theta}, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}.$$

Type of flow	flow field $\mathbf{u}$	potential ϕ	coordinate system
Uniform stream aligned with axis of symmetry	$U\hat{\mathbf{z}}$	Uz	cylindrical
Stagnation point flow at origin	$u_r = \frac{k}{2}r$ $u_z = -kz$	$\frac{k}{4}(r^2 - 2z^2)$	cylindrical
Point source, strength m, at $r = 0$	$\frac{m}{4\pi r^2}\hat{\mathbf{r}}$	$-\frac{m}{4\pi r}$	spherical
Dipole, strength μ, in $\hat{\mathbf{z}}$-direction	$-\frac{\mu}{4\pi r^3}\left(\hat{\mathbf{z}} - 3\frac{z}{r}\hat{\mathbf{r}}\right)$	$-\mu\frac{\cos\theta}{4\pi r^2}$	spherical