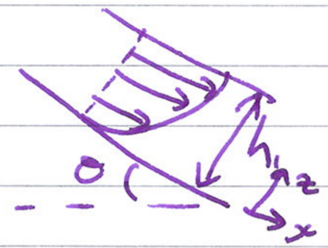
Advanced Fluid Dynamics 2012

similar to worksheet example

1(a)



$\underline{u} = u(z)\hat{\underline{x}}$ so $\nabla\cdot\underline{u} = 0$ automatically

Navier Stokes: $0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} + \rho g\sin\theta$

$0 = -\frac{\partial p}{\partial z} - \rho g\cos\theta$

Hence pressure $p = \rho g\cos\theta(h-z)$ satisfying $p(h)=0$

and $\mu\frac{\partial^2 u}{\partial z^2} = -\rho g\sin\theta$ $\quad u(0)=0$ ~~no~~ no slip

$\frac{du}{dz}(h) = 0$ no shear stress

$\Rightarrow \quad u = \frac{g\sin\theta}{\nu}(h_1 z - z^2/2)$

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Volume flux per unit width $q = \int_0^{h_1} u\,dz = \frac{g\sin\theta h_1^3}{3\nu}$

unseen, but straightforward

(b) Dissipation:

Either $D = 2\mu\int_0^{h_1} e_{ij}e_{ij}\,dz$ where $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

Here $e_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 0 & du/dz \\ 0 & 0 & 0 \\ du/dz & 0 & 0\end{pmatrix}$ $\quad e_{ij}e_{ij} = \frac{1}{2}\left(\frac{du}{dz}\right)^2$

So $D = \mu\int_0^{h_1}\left(\frac{du}{dz}\right)^2 dz = \mu\int_0^{h_1}\left(\frac{g\sin\theta}{\nu}\right)^2(h_1-z)^2\,dz$

$= \mu\left(\frac{g\sin\theta}{\nu}\right)^2\cdot\frac{h_1^3}{3} = \rho g\sin\theta\, q$

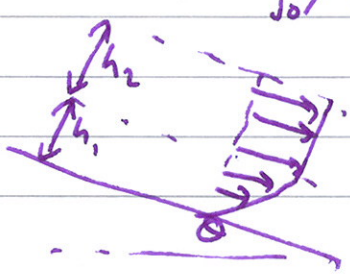
Alternatively: $0 = \mu u\frac{d^2 u}{dz^2} + \rho g\sin\theta\, u$

then $0 = \mu\int_0^{h_1} u\frac{d^2u}{dz^2}dz + \rho g\sin\theta\, q$

$\Rightarrow \quad \int_0^{h_1}\mu\left(\frac{du}{dz}\right)^2 dz = \rho g\sin\theta\, q$

5

(c)



Boundary conditions

$u = 0 \quad z = 0 \quad$ (no-slip)

$\frac{du}{dz} = 0 \quad z = h_1 + h_2 \quad$ (no-stress)

$u(z=h_1^-) = u(z=h_1^+)$ continuity of velocity at $z=h_1$

$\nu \frac{\partial u}{\partial z}(z=h_1^-) = \beta\nu \frac{\partial u}{\partial z}(z=h_1^+)$ continuity of shear stress at $z=h_1$

In $0<z<h_1$ $\qquad 0 = \frac{g\sin\theta}{\nu} + \frac{\partial^2 u_1}{\partial z^2}$

In $h_1<z<h_1+h_2$ $\qquad 0 = \frac{g\sin\theta}{\beta\nu} + \frac{\partial^2 u_2}{\partial z^2}$

So
$$u_1 = \frac{g\sin\theta}{\nu}\left(Az - \tfrac{1}{2}z^2\right) \quad \text{satisfying } u_1(0)=0$$
$$u_2 = \frac{g\sin\theta}{\beta\nu}\left(B + (h_1+h_2)z - \tfrac{1}{2}z^2\right) \quad \text{satisfying } \frac{\partial u_2}{\partial z}=0 \text{ at } z=h_1+h_2$$

Continuity of u : $\quad \frac{g\sin\theta}{\nu}\left(Ah_1 - \frac{h_1^2}{2}\right) = \frac{g\sin\theta}{\beta\nu}\left(B + \frac{h_1^2}{2} + h_1h_2\right)$

Continuity of stress : $\quad g\sin\theta(A-h_1) = g\sin\theta(h_1+h_2-h_1)$

$$\Rightarrow \quad A = h_1+h_2 \quad \text{and} \quad B = \beta\left(\frac{h_1^2}{2}+h_1h_2\right) - \frac{h_1^2}{2} - h_1h_2$$

Flux $\quad q_1 = \int_0^{h_1} u_1\,dz = \frac{g\sin\theta}{\nu}\left[A\frac{z^2}{2} - \frac{z^3}{3}\right]_0^{h_1} = \frac{g\sin\theta}{\nu}\left(\frac{h_1^3}{3} + \frac{h_1^2h_2}{2}\right)$

Flux $\quad q_2 = \int_{h_1}^{h_1+h_2} u_2\,dz = \frac{g\sin\theta}{\beta\nu}\left[Bz + (h_1+h_2)\frac{z^2}{2} - \frac{z^3}{6}\right]_{h_1}^{h_1+h_2}$

$$= \frac{g\sin\theta}{\beta\nu}\Big[(\beta-1)\left(\frac{h_1^2}{2}+h_1h_2\right)h_2 + (h_1+h_2)\left((h_1+h_2)^2-h_1^2\right) - \frac{(h_1+h_2)^3}{6} + h_1^3/6$$

Total volume flux $\quad Q = q_1+q_2 = \frac{g\sin\theta}{3\nu}h_1^3\left(1 + \frac{3}{2}\frac{h_2}{h_1} + \frac{3h_2}{h_1} + 3\frac{h_2^2}{h_1^2} + \frac{1}{\beta}\left(\frac{h_2}{h_1}\right)^3\right)$

$$= \frac{g\sin\theta}{3\nu}h_1^3\left(1 + 3\frac{h_2}{h_1} + 3\left(\frac{h_2}{h_1}\right)^2 + \frac{1}{\beta}\left(\frac{h_2}{h_1}\right)^3\right).$$

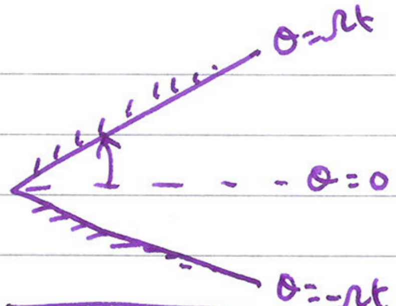
2

(c) The interfacial shear stress is $\rho g\sin\theta h_2$. Hence the bottom layer is driven by this + down slope gravitational acceleration and so is not affected by viscosity of overlying layer.

~~(d) Dissipation~~ $\cancel{= \int_0^{h_1+h_2} \rho g\sin\theta\, u\,dz = \rho g\sin\theta Q}$ $\boxed{\text{omit}}$

~~[Alternative :~~ $\cancel{\int_0^{h_1}\mu\left(\frac{\partial u_1}{\partial z}\right)^2 + \int_{h_1}^{h_1+h_2}\beta\mu\left(\frac{\partial u_2}{\partial z}\right)^2 dz = \rho g\sin\theta Q}$.]

2.



Similar to worksheet

(Polars) ~~Radii~~ $\underline{u} = u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}}$

(b) Boundary conditions on plates:

No tangential velocity $u_r = 0$

Match normal velocity $u_\theta = \pm r\Omega$

on $\theta = \pm \Omega t$.

3 $\boxed{\text{* Interchange (a) \& (b)}}$

In lectures

(a) Governing equations (negligible inertia) $\nabla \cdot \underline{u} = 0 \quad 0 = -\nabla p + \mu \nabla^2 \underline{u}$

Take curl of momentum: $0 = \nabla \wedge (\mu \nabla^2 \underline{u}) = \mu \nabla^2 (\nabla \wedge \underline{u})$

For 2-D motion $\underline{u} = \nabla \wedge \psi \hat{\underline{z}}$ $\quad \hat{\underline{z}}$ = unit vector $\perp^r$ to motion.

Then
$$(\nabla \wedge \underline{u})_i = (\nabla \wedge (\nabla \wedge \psi \hat{\underline{z}}))_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{kpq} \frac{\partial}{\partial x_p} \psi \delta_{q3}$$
$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_p} \psi \delta_{q3}$$
$$= \frac{\partial}{\partial x_3}\frac{\partial \psi}{\partial x_i} - \nabla^2 \psi \delta_{i3} = -\nabla^2 \psi \hat{\underline{z}}$$

Thus $\nabla^2 \nabla \wedge \underline{u} = 0 \quad \Rightarrow \quad \nabla^4 \psi = 0$. $\quad$ Also note $\nabla \cdot \underline{u} = 0$.

Method in lectures - different BCs here.

(c) In terms of streamfunction, $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad u_\theta = -\frac{\partial \psi}{\partial r}$

Boundary conditions demand $\frac{\partial \psi}{\partial \theta} = 0 \quad \frac{\partial \psi}{\partial r} = \mp r\Omega$ on $\theta = \pm \Omega t$.

So seek solution proportional to r^2: $\quad \psi = f(\theta) r^2$.

$$\nabla^2 \psi = 2f + 2f + f'' \qquad \nabla^4 \psi = \frac{1}{r^2}(f'''' + 4f'') = 0$$

Hence
$$f = A\cos 2\theta + B\sin 2\theta + C\theta + D. \qquad A, B, C, D \text{ constants}$$
$$\psi = r^2(A\cos 2\theta + B\sin 2\theta + C\theta + D).$$

$$\frac{\partial \psi}{\partial r} = 2r(A\cos 2\theta + B\sin 2\theta + C\theta + D) = \mp r\Omega \quad \text{on } \theta = \pm \Omega t$$
$$\Rightarrow \quad A\cos 2\Omega t + B\sin 2\Omega t + C\Omega t + D = -\frac{\Omega}{2}$$
and
$$A\cos 2\Omega t - B\sin 2\Omega t - C\Omega t + D = \frac{\Omega}{2}$$

$$\frac{\partial \psi}{\partial \theta} = r^2\left(-2A\cancel{\cos}\sin 2\theta + 2B\cos 2\theta + C\right) = 0 \qquad \theta = \pm\Omega t$$

$$\Rightarrow \qquad -2A\sin 2\Omega t + 2B\cos 2\Omega t + C = 0$$

and $$2A\sin 2\Omega t + 2B\cos 2\Omega t + C = 0$$

Here $A = 0$, $D = 0$ and

$$B\sin 2\Omega t + C\Omega t = -\Omega/2$$
$$2B\cos 2\Omega t + C = 0$$

$$\Rightarrow \quad B\left(\sin 2\Omega t - 2\Omega t\cos 2\Omega t\right) = -\frac{\Omega}{2}$$

$$\Rightarrow \quad B = \frac{-\Omega}{2\left(\sin 2\Omega t - 2\Omega t\cos 2\Omega t\right)}$$

and $$C = \frac{+2\Omega\cos 2\Omega t}{2\left(\sin 2\Omega t - 2\Omega t\cos 2\Omega t\right)}$$

Thus streamfunction $$\psi = -\frac{1}{2}\Omega r^2\left(\frac{\sin 2\theta - 2\theta\cos 2\Omega t}{\sin 2\Omega t - 2\Omega t\cos 2\Omega t}\right)$$

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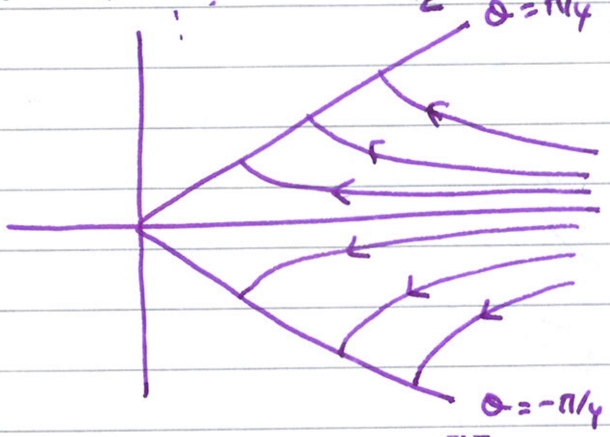
When $\Omega t = \pi/4$ $\qquad \psi = -\frac{1}{2}\Omega r^2 \sin 2\theta$



Streamlines $\psi = \text{const}$

$\Rightarrow r\cos\theta . r\sin\theta = \text{const}$

$\Rightarrow xy = \text{const}.$

2

(d) Navier Stokes equations: $\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$

$$\underline{u}.\nabla\underline{u} \sim \frac{(\Omega r)^2}{r} \qquad \nu\nabla^2\underline{u} \sim \frac{\nu\Omega r}{r^2}$$

unseen

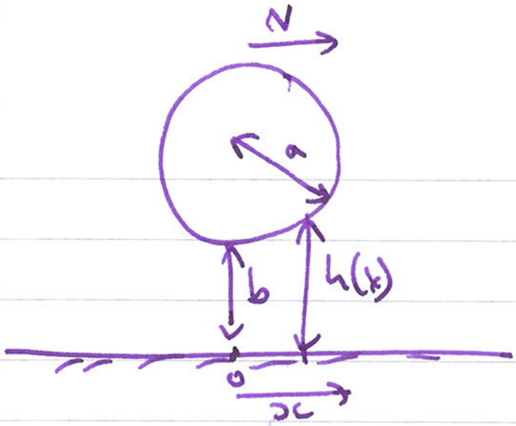
Inertial terms become comparable with viscous when $\Omega^2 r \sim \frac{\nu\Omega}{r}$

$$\Rightarrow \quad r^2 \sim \frac{\nu}{\Omega} \quad \text{ie} \quad r \sim \left(\frac{\nu}{\Omega}\right)^{1/2}$$

The viscous solution is valid for $r \ll \left(\frac{\nu}{\Omega}\right)^{1/2}$

6

3.



Similar to lectures

5

(a) $x^2 + (a-(h-b))^2 = a^2$

$$\Rightarrow \quad h = a + b - (a^2 - x^2)^{1/2}$$
$$= a + b - a\left(1 - x^2/a^2\right)^{1/2}$$
$$\simeq b + \tfrac{1}{2}\, x^2/a + \dots$$

(b) Lubrication regime $(b/a \ll 1)$

Method shown in lectures for other examples

$$0 = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial z^2} \quad \text{subject to} \quad u(0) = -V, \quad u(h) = 0$$

$$\Rightarrow \quad u = \frac{1}{2\mu}\frac{dp}{dx} z^2 + Az + B$$

$u(0) = -V \Rightarrow B = -V$, $\quad u(h) = 0 = \frac{1}{2\mu}\frac{dp}{dx}h^2 + Ah - V$

$$\Rightarrow \quad A = -\frac{1}{2\mu}\frac{dp}{dx}h + \frac{V}{h}$$

$$\Rightarrow \quad u = \frac{1}{2\mu}\frac{dp}{dx}(z^2 - hz) + \frac{Vz}{h} - V$$

Volume flux $\displaystyle q = \int_0^h u\, dz = -\frac{h^3}{12\mu}\frac{dp}{dx} - \frac{Vh}{2}$

But q = constant $\quad \Rightarrow \quad \displaystyle \frac{dp}{dx} = -\frac{12\mu}{h^3}\left(q + \frac{Vh}{2}\right).$

There is no imposed pressure gradient so $\displaystyle \int_{-\infty}^{\infty} \frac{dp}{dx}\, dx = 0$

$$\Rightarrow \quad q\int_{-\infty}^{\infty} \frac{1}{h^3}\, dx = -\frac{V}{2}\int_{-\infty}^{\infty} \frac{1}{h^2}\, dx.$$

$$\int_{-\infty}^{\infty} \frac{1}{h^3}\, dx = \int_{-\infty}^{\infty} \frac{1}{b^3}\sqrt{2ab}\, \frac{1}{(1+s^2)^3}\, ds = \frac{\sqrt{2ab}}{b^3}\cdot\frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \frac{1}{h^2}\, dx = \frac{1}{b^2}\sqrt{2ab}\int_{-\infty}^{\infty} \frac{1}{(1+s^2)^2}\, ds = \frac{\sqrt{2ab}}{b^2}\cdot\frac{\pi}{2}$$

from hint

Thus $\displaystyle \quad q = -\frac{V}{2}\cdot\frac{\sqrt{2ab}}{b^2}\,\frac{\pi}{2}\cdot\frac{8}{3\pi}\,\frac{b^3}{\sqrt{2ab}} = -\frac{2}{3}Vb$

Thus $\displaystyle \quad \frac{dp}{dx} = -\frac{12\mu}{h^3}\left(-\frac{2}{3}Vb + \frac{Vh}{2}\right) = \frac{\mu V}{h^2}\left(8\frac{b}{h} - 6\right)$

10

Method in lectures, but this example is unseen

(c) shear stress on lower boundary $= \mu \frac{\partial u}{\partial z}\Big|_{z=0}$

$$= \mu\left(-\frac{h}{2\mu}\frac{dp}{dx} + \frac{V}{h}\right)$$

$$= \mu\left(\frac{-V}{2h}\left(\frac{8b}{h}-6\right) + \frac{V}{h}\right) = \mu V\left(\frac{4}{h} - \frac{4b}{h^2}\right)$$

Force per unit width $F = \int_{-\infty}^{\infty} \mu \frac{\partial u}{\partial z}\Big|_{z=0} dx = \mu V 4 \int_{-\infty}^{\infty} \frac{1}{h} - \frac{b}{h^2}\, dx$

$\int_{-\infty}^{\infty} \frac{1}{h}\, dx = \frac{1}{b}\sqrt{2ab}\,.\pi$ [from hint)

Thus $F = 4\mu V.\left(\frac{\pi}{b}\sqrt{2ab} - \frac{\sqrt{2ab}}{b}\frac{\pi}{2}\right) = 2\pi \mu \frac{V}{b}\sqrt{2ab}$.

Drag on cylinder $= -F$ because in absence of inertia $\int_S \sigma_{ij} n_j \, dS = 0$
and here $S = \{\text{surface of cylinder}\} \cup \{\text{plane}\}$

10 Thus force F must be applied to maintain the motion.

[* Sufficient to appeal to Newton's III law]

4 (a) $\mathcal{F} = \int_{-\infty}^{\infty} u^2 r \, dz$

$$\frac{d\mathcal{F}}{dr} = \int_{-\infty}^{\infty} u^2 + 2ru\frac{\partial u}{\partial r}\, dz$$

$$= \int_{-\infty}^{\infty} u^2 + 2r\left(\nu \frac{\partial^2 u}{\partial z^2} - w\frac{\partial u}{\partial z}\right) dz$$

$$= \int_{-\infty}^{\infty} u^2 + 2r\left(\nu \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial z}(uw) + u\frac{\partial w}{\partial z}\right) dz \quad \text{but } \frac{\partial w}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(ru)$$

$$= \int_{-\infty}^{\infty} u^2 + 2r\left(\nu \frac{\partial^2 u}{\partial z^2} - \frac{\partial (uw)}{\partial z}\right) - 2ru\frac{1}{r}\frac{\partial}{\partial r}(ru)\, dz$$

$$= \int_{-\infty}^{\infty} 2r\left(\nu \frac{\partial^2 u}{\partial z^2} - \frac{\partial (uw)}{\partial z}\right) - \frac{d\mathcal{F}}{dr}$$

$$\Rightarrow \quad \frac{d\mathcal{F}}{dr} = r\left[\nu\frac{\partial u}{\partial z} - uw\right]_{-\infty}^{\infty} = 0 \quad \text{as } \frac{\partial u}{\partial z}, u \to 0 \text{ as } |z| \to \infty$$

6 Hence $\mathcal{F}$ = constant.

2-D case is in lectures

(b) Scaling: $u^2 r\delta \sim \mathcal{F}$ and from Navier-Stokes $\frac{u^2}{r} \sim \frac{\nu u}{\delta^2}$

$$\Rightarrow \quad u \sim \frac{\nu r}{\delta^2} \quad \text{and} \quad \left(\frac{\nu r}{\delta^2}\right)^2 r\delta \sim \mathcal{F}$$

$$\text{Hence} \quad \delta^3 \sim \frac{\nu^2}{\mathcal{F}} r^3$$

$$\Rightarrow \quad \delta \sim \left(\frac{\nu^2}{\mathcal{F}}\right)^{1/3} r \quad \text{and} \quad u \sim \frac{\nu r}{\left(\frac{\nu^2}{\mathcal{F}}\right)^{2/3} r^2} \sim \left(\frac{\mathcal{F}^2}{\nu}\right)^{1/3}\frac{1}{r}$$

In terms of streamfunction $u = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \Rightarrow \Psi \sim ur\delta$

$$\Rightarrow \Psi \sim \left(\frac{\mathcal{F}^2}{\nu}\right)^{1/3}\left(\frac{\nu^2}{\mathcal{F}}\right)^{1/3} r = (\mathcal{F}\nu)^{1/3} r.$$

6 Hence pose $\delta(r) = \left(\frac{\nu^2}{\mathcal{F}}\right)^{1/3} r$ and $\Psi = -(\mathcal{F}\nu)^{1/3} r f\left(\frac{z}{\delta}\right)$

2 "Thin" implies $\delta/r \ll 1 \Rightarrow (\nu^2/\mathcal{F})^{1/3} \ll 1$. Reynolds Number $= \frac{(\mathcal{F}\nu)^{1/3}}{\nu} = \left(\frac{\mathcal{F}}{\nu^2}\right)^{1/3}$

Similar to 2-D reasoning

(c) On substitution into momentum equation, gives $f''' + ff'' + f'^2 = 0$

this means Re ≫ 1

Boundary condition: (i) $u \to 0$ as $z \to \infty \Rightarrow f' \to 0$ as $\eta = \frac{z}{\delta} \to \infty$

1

(ii) Symmetry in $z=0$ for velocity field u

$$\Rightarrow \quad f(y) \text{ is odd function} \qquad [f(0)=0,\ f''(0)=0]$$

1

(iii) $$\int_{-\infty}^{\infty} u^2 r\,dz = \mathcal{F} \Rightarrow \int_{-\infty}^{\infty}\left(\frac{-1}{r}\frac{\partial \Psi}{\partial z}\right)^2 r\,\delta\,dy = \mathcal{F}$$

but $$\frac{\partial \Psi}{\partial z} = -(\mathcal{F}\nu)^{1/3}\frac{r}{\delta}f'$$

$$\Rightarrow \int_{-\infty}^{\infty}(\mathcal{F}\nu)^{2/3}\frac{1}{\delta^2}r\,\delta\,f'^2\,dy = \mathcal{F}$$

but $$r/\delta = \left(\frac{\mathcal{F}}{\nu^2}\right)^{1/3} \Rightarrow \int_{-\infty}^{\infty} f'^2\,dy = 1.$$

2

Unseen, but similar to 2-D

(d) Integrating $$f''' + ff'' + f'^2 = 0$$

$$f'' + ff' = \text{const} = 0$$

integrating $$f' + \tfrac{1}{2}f^2 = \tfrac{1}{2}c^2 \qquad (\text{assuming } f \to c \text{ as } y \to \infty)$$

$$\int \frac{df}{\frac{1}{2}(c^2-f^2)} = \int dy$$

$f = c\tanh s$ $$\int \frac{2c\,\mathrm{sech}^2 s\,ds}{c^2(1-\tanh^2 s)} = \int dy$$

$$\Rightarrow \frac{2}{c}\tanh^{-1}\left(\frac{f}{c}\right) = y + c_1$$ but $c_1 = 0$ as $f(y)$ is odd

$$\Rightarrow \quad f = c\tanh\frac{cy}{2}.$$

To determine c: $$\frac{df}{dy} = \frac{c^2}{2}\mathrm{sech}^2\frac{cy}{2} \qquad \frac{c^4}{4}\int_{-\infty}^{\infty}\mathrm{sech}^4\frac{cy}{2}\,dy = 1$$

$p = \frac{cy}{2}$ $$\Rightarrow \frac{c^3}{2}\int_{-\infty}^{\infty}\mathrm{sech}^4 p\,dp = 1$$

$$\int_{-\infty}^{\infty}\mathrm{sech}^4 p\,dp = \int_{-\infty}^{\infty}\mathrm{sech}^2 p(1-\tanh^2 p)\,dp = \left[\tanh p - \tfrac{1}{3}\tanh^3 p\right]_{-\infty}^{\infty} = 4/3$$

5 $$\Rightarrow c^3 = 3/2 \quad \text{and} \quad f = (3/2)^{1/3}\tanh\left[\left(\tfrac{3}{2}\right)^{1/3}\frac{y}{2}\right]$$

$$rW = \frac{\partial \Psi}{\partial r} \to -\left(\frac{3\mathcal{F}\nu}{2}\right)^{1/3} \text{ as } |z/\delta| \to \infty.$$ Non zero because jet entrains fluid.

2

Unseen, but essentially identical to 2-D case.

5. The undisturbed flow $\underline{u} = U_0(z)\hat{\underline{x}}$, $p = p_0$

(a) Introduce perturbation $\varepsilon \underline{u}_1 \equiv \varepsilon(u_1, 0, w_1)$ [2D perturbation]

$$\nabla \cdot \underline{u}_1 = 0 \quad \Rightarrow \quad \frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p \quad \Rightarrow \quad \rho\left(\frac{\partial u_1}{\partial t} + U_0 \frac{\partial u_1}{\partial x} + w_1 \frac{dU_0}{dz}\right) = -\frac{\partial p_1}{\partial x}$$

$$\rho\left(\frac{\partial w_1}{\partial t} + U_0 \frac{\partial w_1}{\partial x}\right) = -\frac{\partial p_1}{\partial z}$$

Linearised equations

$$-\nabla^2 p_1 = \rho\left(\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z}\right) + U_0 \frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z}\right) + \frac{\partial w_1}{\partial x}\frac{dU_0}{dz} + \frac{dU_0}{dz}\frac{\partial w_1}{\partial x}\right)$$
$$= 2\rho \frac{dU_0}{dz}\frac{\partial w_1}{\partial x}.$$

Then
$$\rho \frac{\partial}{\partial t}\nabla^2 w_1 + \rho U_0 \frac{\partial \nabla^2 w_1}{\partial x} + 2\rho \frac{dU_0}{dz}\frac{\partial^2 w_1}{\partial x \partial z} + \rho \frac{d^2 U_0}{dz^2}\frac{\partial w_1}{\partial x}$$
$$= -\frac{\partial}{\partial z}(\nabla^2 p_1)$$
$$= +2\rho \frac{dU_0}{dz}\frac{\partial^2 w_1}{\partial x \partial z} + 2\rho \frac{d^2 U_0}{dz^2}\frac{\partial w_1}{\partial x}.$$

Hence
$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)\nabla^2 w_1 - \frac{d^2 U_0}{dz^2}\frac{\partial w_1}{\partial x} = 0.$$

In lectures 6

(b) Introduce
$$w_1 = (U_0 - c)\phi\, e^{ik(x-ct)}$$
$$\frac{\partial w_1}{\partial x} = ik(U_0 - c)\phi\, e^{ik(x-ct)}$$
$$\nabla^2 w_1 = \left[\frac{d^2}{dz^2}(U_0 - c)\phi - k^2 (U_0 - c)\phi\right] e^{ik(x-ct)}$$

Thus
$$ik(U_0 - c)\left[(U_0 - c)\phi'' + 2U_0'\phi' + U_0''\phi - k^2(U_0 - c)\phi\right] - ikU_0''(U_0 - c)\phi = 0$$

$$\Rightarrow (U_0 - c)^2\phi'' + 2U_0'(U_0 - c)\phi' - k^2(U_0 - c)^2\phi = 0$$

$$\Rightarrow \frac{d}{dz}\left((U_0 - c)^2 \frac{d\phi}{dz}\right) - k^2 (U_0 - c)^2 \phi = 0$$

similar to lectures 4

(c)
$$\int_0^d \phi^* \frac{d}{dz}\left((U_0 - c)^2 \frac{d\phi}{dz}\right) dz = \left[\phi^* (U_0 - c)^2 \frac{d\phi}{dz}\right]_0^d - \int_0^d (U_0 - c)^2 \left|\frac{d\phi}{dz}\right|^2 dz$$

but $\phi^* = 0$ at $z = 0, d$ $\Rightarrow$ $\displaystyle\int_0^d (U_0 - c)^2 Q\, dz = 0$

where $Q = \left|\frac{d\phi}{dz}\right|^2 + k^2|\phi|^2$

Real part is unseen

Introduce $c = c_r + i c_i$ $\qquad (u_0 - c)^2 = ((u_0 - c_r)^2 - c_i^2) - i2c_i(u_0 - c_r)$

Taking real part : $\displaystyle\int_0^d ((u_0 - c_r)^2 - c_i^2) Q \, dz = 0$

6 $\quad$ imaginary part : $\displaystyle c_i \int_0^d (u_0 - c_r) Q \, dz = 0.$

(d) $\max\{u_0\} = u_1$, $\min\{u_0\} = u_2$ then $(u_0 - u_1)(u_0 - u_2) \leq 0$

Unseen

Hence $\displaystyle\int_0^d (u_0 - u_1)(u_0 - u_2) Q \, dz \leq 0$

If flow is unstable then $\displaystyle\int_0^d (u_0 - c_r) Q \, dz = 0$ as $c_i > 0$

2

Expanding $(u_0 - u_1)(u_0 - u_2) = u_0^2 - u_0(u_1 + u_2) + u_1 u_2$

then $\displaystyle\int_0^d u_0(u_1 + u_2) Q \, dz = \int_0^d c_r (u_1 + u_2) Q \, dz$

and $\displaystyle\int_0^d u_0^2 Q \, dz = \int_0^d (c_i^2 + 2u_0 c_r - c_r^2) Q \, dz$

$\displaystyle\qquad = \int_0^d (c_i^2 + c_r^2) Q \, dz.$

4

Thus $\displaystyle\int_0^d (c_r^2 + c_i^2 - (u_1 + u_2) c_r + u_1 u_2) Q \, dz \leq 0$

As $Q \geq 0$ this implies $c_r^2 + c_i^2 - (u_1 + u_2) c_r + u_1 u_2 \leq 0$

3 $\Rightarrow$ $\displaystyle\left(c_r - \frac{(u_1 + u_2)}{2}\right)^2 + c_i^2 \leq \left(\frac{u_1 + u_2}{2}\right)^2 - u_1 u_2$

$\displaystyle\qquad = \left(\frac{u_1 - u_2}{2}\right)^2$