Advanced Fluid Dynamics

1(a) $\underline{u} = u(z,t)\hat{\underline{x}}$ $\quad \rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial z^2}$ $\quad u(0,t) = U\cos\omega t$

$u \to 0$ as $z \to \infty$

write $u = \mathrm{Re}\{f(z)e^{i\omega t}\}$

then $f'' - \delta^2 f = 0$ where $\delta^2 = \frac{i\rho\omega}{\mu} \to \delta = \left(\frac{\rho\omega}{\mu}\right)^{1/2} e^{i\pi/4}$

Applying the boundary conditions, $f = U\exp(-\delta z)$

Velocity field $u = \mathrm{Re}\{U e^{-\delta z + i\omega t}\}$

3 $\quad = U e^{-\left(\frac{\rho\omega}{2\mu}\right)^{1/2} z} \cos\left(\omega t - \left(\frac{\rho\omega}{2\mu}\right)^{1/2} z\right)$

Shear stress at boundary $\tau_{xz}(0) = \mu \frac{du}{dz}(0)$

$= \mathrm{Re}\{-\mu\delta U e^{i\omega t}\}$

$\to \tau_{xz}(0) = -\mathrm{Re}\{\cancel{\delta}\mu U \left(\frac{\rho\omega}{\mu}\right)^{1/2} e^{i(\omega t + \pi/4)}\}$

3 $\quad = -\mu U\lambda \cos(\omega t + \pi/4)$ where $\lambda = \left(\frac{\rho\omega}{\mu}\right)^{1/2}$

Standard example : similar to lectures

(b) Rate of working at boundary $= u(0)\tau_{xz}(0)$

Average rate of working $= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} U\cos\omega t . \mu U\lambda \cos(\omega t + \frac{\pi}{4})\,dt$

$\int_0^{2\pi/\omega} \cos\omega t \cos(\omega t + \frac{\pi}{4})\,dt = \int_0^{2\pi/\omega} \cos\omega t(\cos\omega t - \sin\omega t)\frac{1}{\sqrt{2}}\,dt$

$= \frac{1}{\sqrt{2}}\int_0^{2\pi/\omega} \frac{1}{2}(1 + \cos 2\omega t) - \frac{1}{2}\sin 2\omega t\,dt$

$= \frac{1}{\sqrt{2}} . \frac{1}{2} . {}^{2\pi}/_{\omega}$

Hence Average rate of working $= \frac{1}{2\sqrt{2}} . \mu U^2 \lambda = \frac{1}{2\sqrt{2}}\mu U^2 \left(\frac{\rho\omega}{\mu}\right)^{1/2}$

5 $\quad$ = Average rate of dissipation (balance of energy)

In lectures, evaluated using ~~dif~~ different method

(c) Now solve $f'' - \delta^2 f = 0$ subject to $f(0) = U$ and $f(d) = U$

$$f = A \sinh \delta z + B \cosh \delta z$$

$f(0) = U \Rightarrow B = U$, $\quad f(d) = U = A \sinh \delta d + B \cosh \delta d$

$$\Rightarrow A = \frac{U(1-\cosh \delta d)}{\sinh \delta d}$$

$$\Rightarrow \quad f = \frac{U \sinh \delta z\,(1-\cosh\delta d)}{\sinh \delta d} + U\cosh \delta z$$

$$= U\left[\frac{\sinh \delta z}{\sinh \delta d} + \frac{\sinh(\delta(d-z))}{\sinh \delta d}\right]$$

$$= \frac{U}{\sinh \delta d}\left(2 \sinh \frac{\delta d}{2} \cosh\left(\frac{\delta(d-2z)}{2}\right)\right)$$

$$= U \cdot \frac{\cosh\left[\delta\left(\frac{d}{2}-z\right)\right]}{\cosh\left[\frac{\delta d}{2}\right]}$$

Hence velocity $\quad u = \mathrm{Re}\left\{ U \frac{\cosh\left[\delta\left(\frac{d}{2}-z\right)\right]}{\cosh\left[\frac{\delta d}{2}\right]} e^{i\omega t}\right\}$

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Unseen

~~Shear stress at $z=0$ $\tau_{xz} = \mathrm{Re}\left\{-\mu\delta U \tanh\left(\frac{\delta d}{2}\right) e^{i\omega t}\right\}$~~

~~$= -\mu\lambda U\ \mathrm{Re}\left\{\tanh\left(\frac{\delta d}{2}\right) e^{i(\omega t+\pi/4)}\right\}$~~

Vanishes on centreline if $\mathrm{Re}\left\{\frac{U}{\cosh\left[\frac{\delta d}{2}\right]} e^{i\omega t}\right\} = 0$

$$\Rightarrow \quad \mathrm{Re}\left\{\frac{1}{\cosh\left[\delta d/2\right]}\right\} = 0$$

but $\delta = \lambda \frac{d}{2}\frac{1}{\sqrt{2}}(1+i)$ and $\cosh[M(1+i)] = \cosh M \cos M + i \sinh M \sin M$

3 Hence require $\cos\left(\frac{\lambda d}{2\sqrt{2}}\right) = 0 \Rightarrow \lambda d = 2\sqrt{2}\left(\frac{1}{2} + n\pi\right)$ $n \in \mathbb{Z}$

(d) Shear stress $\tau_{xz} = \mathrm{Re}\left\{\mu \frac{\partial u}{\partial z}\Big|_{z=0}\right\} = -\mu\lambda U\ \mathrm{Re}\left\{\tanh\left(\frac{\delta d}{2}\right) e^{i(\omega t+\pi/4)}\right\}$

Unseen

When $\lambda d \gg 1 \qquad \tau_{xz} = -\mu\lambda U \cos(\omega t + \pi/4)$

When $\lambda d \ll 1 \qquad \tau_{xz} = -\mu\lambda U\ \mathrm{Re}\left\{\frac{\lambda d}{2} e^{i(\omega t+\frac{\pi}{2})}\right\} = -\mu\frac{\lambda^2 d U}{2}\cos\left(\omega t + \frac{\pi}{2}\right)$

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similar to lectures

2.(a) $\underline{u} = \Gamma z \hat{\underline{x}}$ $\quad \underline{u} = \begin{pmatrix}0&0&\Gamma/2\\0&0&0\\\Gamma/2&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} + \begin{pmatrix}0&0&\Gamma/2\\0&0&0\\-\Gamma/2&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$

__Note__: if $\underline{\Omega} = (0, -\frac{\Gamma}{2}, 0)$ $\quad \underline{x}\wedge\underline{\Omega} = (\Gamma/2, 0, -\Gamma/2)$

So symmetric tensor $E_{ij} = \begin{pmatrix}0&0&\Gamma/2\\0&0&0\\+\Gamma/2&0&0\end{pmatrix}$ $\quad \underline{\Omega} = (0, -\frac{\Gamma}{2}, 0)$

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[Alternative: $\nabla\wedge(\underline{x}\wedge\underline{\Omega}) = -2\underline{\Omega}$ and $\nabla\wedge(\frac{\Gamma}{2}z, 0, -\frac{\Gamma}{2}x) = (0, \Gamma, 0)$]

Method similar to example in lectures, but this is otherwise unseen

(b) $u_i = (\underline{x}\wedge\underline{\Omega})_i\left(1-\frac{a^3}{r^3}\right) + E_{ij}x_j\left(1-\frac{a^5}{r^5}\right) - \frac{5}{2}x_pE_{pq}x_q\left(\frac{a^3}{r^5}-\frac{a^5}{r^7}\right)x_i$

$$\nabla_j u_i = \varepsilon_{ijq}\Omega_q\left(1-\frac{a^3}{r^3}\right) + \frac{3a^3}{r^5}\varepsilon_{ipq}x_p\Omega_q x_j$$
$$+ E_{ij}\left(1-\frac{a^5}{r^5}\right) + E_{ip}x_p\frac{5a^5}{r^7}x_j$$
$$-\frac{5}{2}x_pE_{pq}x_q\left(\frac{a^3}{r^5}-\frac{a^5}{r^7}\right)\delta_{ij} + \frac{5}{2}x_i\left(\frac{a^3}{r^5}-\frac{a^5}{r^7}\right)\left(E_{jq}x_q + x_pE_{pj}\right)$$
$$-\frac{5}{2}x_pE_{pq}x_q\left(\frac{-5a^3}{r^7}+\frac{7a^5}{r^9}\right)x_ix_j$$

$$\nabla_j u_i\Big|_{r=a} = +\frac{3}{a^2}\varepsilon_{ipq}x_p\Omega_q x_j + \frac{5}{a^2}E_{ip}x_px_j - \frac{5}{a^4}x_pE_{pq}x_qx_ix_j$$

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Stress tensor $\sigma_{ij} = +p\delta_{ij} + \mu(\nabla_iu_j + \nabla_ju_i)$ on $r=a$

$$= +\frac{5\mu}{a^2}x_pE_{pq}x_q\delta_{ij} + \frac{3\mu}{a^2}\left(\varepsilon_{ipq}x_p\Omega_qx_j + \varepsilon_{jpq}x_p\Omega_qx_i\right)$$
$$+\frac{5\mu}{a^2}\left(E_{ip}x_px_j + E_{jp}x_px_i\right) - \frac{10\mu}{a^4}x_pE_{pq}x_qx_ix_j$$

Shear stress on surface of particle: $\underline{n} = \frac{1}{a}\underline{x}$ $\qquad$ [exerted on particle]

$$\sigma_{ij}n_j = +\frac{5\mu}{a^3}x_pE_{pq}x_qx_i + \frac{3\mu}{a}\varepsilon_{ipq}x_p\Omega_q$$
$$+\frac{5\mu}{a^3}\left(E_{ip}x_pa^2 + x_jE_{jp}x_px_i\right) - \frac{10\mu}{a^5}x_pE_{pq}x_qa^2x_i$$

5 $\quad = +\frac{3\mu}{a}(\underline{x}\wedge\underline{\Omega})_i + \frac{5\mu}{a}E_{ip}x_p.$

Unseen, though familiar method

(c) Drag Force $F = \int_{r=a} \tau_i \, dS$

$$= \int_{r=a} \left(+\frac{3\mu}{a} \varepsilon_{ijk} x_j \Omega_k + \frac{5\mu}{a} E_{ip} x_p\right) dS$$

but $\int_{r=a} x_i \, dS = 0 \Rightarrow$ Drag Force $F = 0$

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(d) Torque exerted $\underline{G} = \int_{r=a} \underline{x} \wedge \underline{\tau} \, dS$

$$\Rightarrow G_i = \int_{r=a} \varepsilon_{ijk} x_j \left(+\frac{3\mu}{a} \varepsilon_{kpq} x_p \Omega_k + \frac{5\mu}{a} E_{kp} x_p\right) dS$$

But $\int_{r=a} x_j x_p \, dS = \frac{4\pi a^4}{3} \delta_{jp}$ from hint

$$\int \varepsilon_{ijk} x_j \frac{5\mu}{a} E_{kp} x_p \, dS = \frac{4\pi a^4}{3} \frac{5\mu}{a} E_{kp} \varepsilon_{ijk} \delta_{jp} = \frac{5\mu}{a} \varepsilon_{ijk} E_{kj} \frac{4\pi a^4}{3}$$

but ε_{ijk} is antisymmetric, E_{ij} is symmetric $\Big\}$ $\varepsilon_{ijk} E_{kj} = 0$

$$\int \varepsilon_{ijk} x_j \frac{+3\mu}{a} \varepsilon_{kpq} x_p \Omega_k \, dS = +\frac{3\mu}{a} \cdot \frac{4\pi a^4}{3} \cdot \varepsilon_{ijk} \delta_{jp} \varepsilon_{kpq} \Omega_k$$

$$= +4\pi\mu a^3 \varepsilon_{ijk} \varepsilon_{kjq} \Omega_k$$

and $\varepsilon_{ijk}\varepsilon_{kjq} = \varepsilon_{kij}\varepsilon_{kjq} = 3\delta_{ij}\delta_{jq} - 3\delta_{iq} = -2\delta_{iq}$

$\Rightarrow$ Torque applied $G_i = -8\pi\mu a^3 \Omega_i$

$$\Rightarrow \underline{G} = 4\pi\mu a^3 (0, \Gamma, 0)$$

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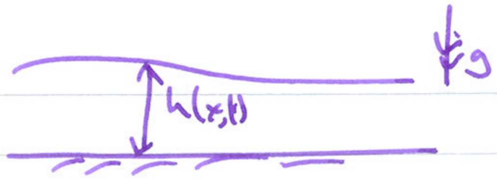
Unseen, though similar method on worksheet

3 (a) Thin layer



Bookwork

Governing equations: $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{U}{L} \sim \frac{W}{H}$

Vertical momentum $\quad \underbrace{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}}_{\frac{U^2 H}{L^2}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\frac{P}{\rho H}} + \underbrace{\nu \nabla^2 w}_{\nu \frac{HU}{LH^2}} + \underbrace{g}_{g}$ —(2)

Horizontal momentum $\quad \underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}}_{\frac{U^2}{L}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\frac{P}{\rho L}} + \underbrace{\nu \nabla^2 u}_{\frac{\nu U}{H^2}}$ —(1)

Assume $\quad U^2/L \ll \nu U/H^2 \quad \Rightarrow \quad \frac{H}{L}\cdot\frac{UH}{\nu} \ll 1$
and $\quad P \sim \rho L \nu U/H^2$
} in —(1)

Then in —(2). $\quad \frac{U^2 H}{L^2}, \quad \frac{\nu L U}{H^3}, \quad \frac{\nu U}{HL}, \quad g$

Balance $\quad \frac{\nu L U}{H^3} \sim g \quad \Rightarrow \quad P \sim gH$ and leading order dynamics

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + g \end{aligned}\right\}$$

6 Hence pressure $\quad p = p_{atm} + \rho g(h-z)$

(b) Pressure gradient $\quad \frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x}$

$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2} = \frac{\rho g}{\rho \nu}\frac{\partial h}{\partial x}$ subject to $u(0) = 0$ (no-slip) $\quad \frac{\partial u}{\partial z} = 0$ at $z = h$ (no-stress)

$\Rightarrow \quad u = \frac{g}{\nu}\frac{\partial h}{\partial x}\left(\frac{1}{2}z^2 - hz\right)$

Flux $\quad q = \int_0^h u \, dz = -\frac{g h^3}{3\nu}\frac{\partial h}{\partial x}$

Mass conservation $\quad w(h) + \int_0^h \frac{\partial u}{\partial x} dz = 0$

$$\Rightarrow \quad w(h) + \frac{\partial}{\partial x}\int_0^h u\, dz - u(h)\frac{\partial h}{\partial x} = 0$$

but kinematic condition at $z=h \rightarrow \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w.$

$$\Rightarrow \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\, dz = 0$$

7 $$\Rightarrow \quad \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\frac{g h^3}{3\nu}\frac{\partial h}{\partial x}\right)$$

In lectures

(c) $h = h_0 + a \qquad a/h_0 \ll 1 \Rightarrow \frac{\partial a}{\partial t} = \frac{g h_0^3}{3\nu}\frac{\partial^2 a}{\partial x^2}$ 3

(for linearisation)

Try $\quad a = \frac{a_0}{f(t)} \exp\left(\frac{-x^2}{g(t)}\right)$

$$\frac{\partial a}{\partial t} = -\frac{a_0}{f^2}\frac{\partial f}{\partial t}\exp\left(\frac{-x^2}{g(t)}\right) + \frac{a_0}{f}\frac{x^2}{g^2}\frac{\partial g}{\partial t}\exp\left(\frac{-x^2}{g(t)}\right)$$

$$\frac{\partial a}{\partial x} = -\frac{2x a_0}{fg}\exp\left(\frac{-x^2}{g}\right)$$

$$\frac{\partial^2 a}{\partial x^2} = \frac{-2a_0}{fg}\exp\left(\frac{-x^2}{g}\right) + \frac{4x^2 a_0}{fg^2}\exp\left(\frac{-x^2}{g}\right)$$

Hence to satisfy $\quad \frac{\partial a}{\partial t} = \frac{g h_0^3}{3\nu}\frac{\partial^2 a}{\partial x^2}$ requires

$$-\frac{1}{f^2}\frac{\partial f}{\partial t} a_0 = -\frac{2a_0}{fg}\cdot\frac{g h_0^3}{3\nu}$$

and $$\frac{1}{fg^2}\frac{\partial g}{\partial t} = \frac{4}{fg^2}\cdot\frac{g h_0^3}{3\nu}$$

Hence $\quad g = \frac{4}{3}\frac{g h_0^3}{\nu} t + c \qquad c = \text{constant}$

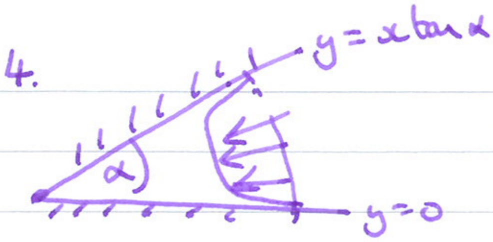
and $$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{\frac{2}{3}\frac{g h_0^3}{\nu}}{\frac{4}{3}\frac{g h_0^3}{\nu}t + c} \Rightarrow \log f = \frac{1}{2}\log\left|\frac{4}{3}\frac{g h_0^3}{\nu}t + c\right| + c_1$$

unseen

$$\Rightarrow a(x,t) = \frac{a_0}{A\left(\frac{4}{3}\frac{g h_0^3}{\nu}t + c\right)^{1/2}}\exp\left(\frac{-x^2}{\left(\frac{4}{3}\frac{g h_0^3}{\nu}t + c\right)}\right) \qquad 9$$

Matching initial condition $\Rightarrow a(x,t) = \frac{a_0}{\left(\frac{4}{3}\frac{g h_0^3}{\nu}t + 1\right)^{1/2}}\exp\left(\frac{-x^2}{\left(\frac{4}{3}\frac{g h_0^3}{\nu}t + 1\right)}\right)$

4.



Inflow to origin.

Similar to lectures

(a) Look for solution $\underline{u} = u_r(r)\,\hat{\underline{r}}$ and $p = p(r)$

$\Rightarrow$ (Mass) $\quad \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0$

and (Momentum) $\quad u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$ (inviscid)

$\Rightarrow \quad u_r = \frac{K}{r} \quad (K = \text{constant})$

but Volume flux $\quad \alpha Q = -\int_0^\alpha r u_r \, d\theta = -K\alpha \Rightarrow K = -Q$

Pressure $\quad \frac{1}{\rho}\frac{\partial p}{\partial r} = -\cdot\frac{-Q}{r}\cdot\frac{Q}{r^2} = \frac{Q^2}{r^3}$

6 $\quad \Rightarrow \quad p = p_0 - \rho\, Q^2/2r^2$

Similar to lectures

(b) This solution does not satisfy no-slip boundary conditions at $y=0$ and $y = x\tan\alpha$ $\Rightarrow$ need to introduce boundary layers.

Reynolds number $\quad Re = \text{"}\frac{U.L}{\nu}\text{"} = \frac{Q}{r}\cdot\frac{r}{\nu} = \frac{Q}{\nu}.$

3

Unseen except but b-L in lectures

(c) Boundary layer at $y=0$: $\psi(x,y)$ $\quad u = \frac{\partial\psi}{\partial y} \quad v = -\frac{\partial\psi}{\partial x}.$

Boundary layer equation $\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \quad \frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \cdot\frac{Q^2}{x^3} + \nu\frac{\partial^3\psi}{\partial y^3}$$

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(d) Look for solution of form $\quad \psi = -\sqrt{\nu Q}\, f\left(\frac{y}{x}\sqrt{\frac{Q}{\nu}}\right)$

$$\frac{\partial\psi}{\partial y} = -\frac{Q}{x}f' \qquad \frac{\partial^2\psi}{\partial y^2} = -Q\sqrt{\frac{Q}{\nu}}\,\frac{1}{x^2}f'' \qquad \frac{\partial^3\psi}{\partial y^3} = -\frac{Q^2}{\nu}\frac{1}{x^3}f'''$$

Unseen example

$$\frac{\partial \psi}{\partial x} = +\sqrt{\nu Q}\cdot\frac{y}{x^2}\sqrt{\frac{Q}{\nu}}\,f' = \frac{Qy}{x^2}f'$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{Q}{x^2}f' + \frac{Qy}{x^2}\sqrt{\frac{Q}{\nu}}\,f''$$

Hence: $-\frac{Q}{x}f'\cdot\left(\frac{Q}{x^2}f' + \frac{Qy}{x^2}\sqrt{\frac{Q}{\nu}}\,f''\right) + \frac{Qy}{x^2}f'\cdot Q\sqrt{\frac{Q}{\nu}}\frac{1}{x^2}f''$

$$= -\frac{Q^2}{x^3} + -\nu\cdot\frac{Q^2}{\nu}\frac{1}{x^3}f'''$$

$$\Rightarrow \quad -f'^2 = -1 - f'''$$

7 $\Rightarrow \quad f''' - f'^2 + 1 = 0 \quad —— (*)$

(e) Boundary conditions:

Unseen

(i) $v=0$ at $y=0$ $\Rightarrow$ $\frac{\partial \psi}{\partial x} = 0$ automatically satisfied

(ii) $u=0$ at $y=0$ $\Rightarrow$ $\frac{\partial \psi}{\partial y} = 0$ $\Rightarrow$ $f'(0)=0$

(iii) $u = -\frac{Q}{x}$ as $y\to\infty$ $\Rightarrow$ $\frac{\partial \psi}{\partial y} \to -\frac{Q}{x}$ $\Rightarrow$ $f' \to 1$ as $\eta = \frac{y}{x}\sqrt{\frac{Q}{\nu}} \to \infty$

3 Only need conditions (ii) & (iii) as —— (*) can be viewed as an ODE for f'.

i.e. if $g = f'$ $\quad g'' + g^2 - 1 = 0 \quad g(0)=0,\ g\to 1$ as $\eta\to\infty$

5. Basic state : $\underline{u} = U_\theta(r)\,\hat{\underline{\theta}}$, $p = P(r)$ satisfying $-U_\theta^2/r = -\frac{1}{\rho}\frac{\partial P}{\partial r}$.

(a) Perturbation $u = U_\theta(r)\,\hat{\underline{\theta}} + \underline{u}'(r,z,t)$ $\quad p = P(r) + p'(r,z,t)$

[Linearised equation]

$$\frac{\partial u_r'}{\partial t} - \frac{2U_\theta u_\theta'}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

$$\frac{\partial u_\theta'}{\partial t} + u_r'\frac{\partial U_\theta}{\partial r} + \frac{u_r' U_\theta}{r} = 0$$

$$\frac{\partial u_z'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r') + \frac{\partial u_z'}{\partial z} = 0$$

similar to 2D except in lectures

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(b) Now write $\underline{u}' = e^{i(kz+\omega t)}(\hat{u}_r, \hat{u}_\theta, \hat{u}_z)$, $p' = \hat{p}\, e^{i(kz+\omega t)}$

$$i\omega\hat{u}_r - \frac{2U_\theta\hat{u}_\theta}{r} = -\frac{1}{\rho}\frac{\partial\hat{p}}{\partial r} \quad \text{— (1)}$$

$$i\omega\hat{u}_\theta + \hat{u}_r\frac{\partial U_\theta}{\partial r} + \frac{\hat{u}_r U_\theta}{r} = 0 \quad \text{— (2)}$$

$$i\omega\hat{u}_z = -\frac{ik}{\rho}\hat{p} \quad \text{— (3)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}_r) + ik\hat{u}_z = 0 \quad \text{— (4)}$$

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From —(3) and —(4) $\quad -\frac{ik}{\rho}\hat{p} = \frac{i\omega}{-ik}\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}_r)$

From —(2) $\quad \hat{u}_\theta = \frac{-1}{i\omega}\frac{1}{r}\hat{u}_r\frac{\partial}{\partial r}(rU_\theta)$

Into —(1): $\quad i\omega\hat{u}_r - \frac{2U_\theta}{r}\cdot\frac{-1}{i\omega}\frac{1}{r}\hat{u}_r\frac{\partial}{\partial r}(rU_\theta) = \frac{1}{ik}\frac{\partial}{\partial r}\left(\frac{\omega}{-k}\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}_r)\right)$

$$\Rightarrow \quad i\omega\hat{u}_r - \frac{i}{\omega}\hat{u}_r\frac{2U_\theta}{r^2}\frac{\partial}{\partial r}(rU_\theta) = \frac{i\omega}{k^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}_r)\right)$$

$$\Rightarrow \quad k^2\hat{u}_r - \frac{k^2}{\omega^2}\frac{\hat{u}_r}{r^3}\frac{\partial}{\partial r}(r^2U_\theta^2) = \frac{\partial}{\partial r}\left(\frac{1}{r}\left(r\frac{\partial\hat{u}_r}{\partial r} + \hat{u}_r\right)\right)$$

$$= \frac{\partial^2\hat{u}_r}{\partial r^2} + \frac{1}{r}\frac{\partial\hat{u}_r}{\partial r} - \frac{\hat{u}_r}{r^2}$$

linear

$$\Rightarrow \quad \left[k^2 + \frac{1}{r^2} - \frac{k^2}{\omega^2 r^3}\frac{\partial}{\partial r}(r^2 u_\theta^2)\right]\hat{u}_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{d\hat{u}_r}{\partial r}\right).$$

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(c) $$\int_{R_1}^{R_2} r\hat{u}_r^2\left[k^2 + \frac{1}{r^2} - \frac{k^2}{\omega^2 r^3}\frac{\partial}{\partial r}(r^2 u_\theta^2)\right]dr = \int_{R_1}^{R_2} u_r \frac{\partial}{\partial r}\left(r\frac{\partial \hat{u}_r}{\partial r}\right)dr$$

but $$\int_{R_1}^{R_2} u_r \frac{\partial}{\partial r}\left(r\frac{d\hat{u}_r}{\partial r}\right)dr = \left[u_r r \frac{d\hat{u}_r}{\partial r}\right]_{R_1}^{R_2} - \int_{R_1}^{R_2} r\left(\frac{d\hat{u}_r}{\partial r}\right)^2 dr$$

Hence $$\int_{R_1}^{R_2} r\left[\hat{u}_r^2\left(k^2+\frac{1}{r^2}\right) + \left(\frac{du_r}{\partial r}\right)^2\right]dr = \frac{k^2}{\omega^2}\int_{R_1}^{R_2}\hat{u}_r^2 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_\theta^2)\,dr$$

$$\Rightarrow \quad \omega^2 \int_{R_1}^{R_2} \Phi \, dr = k^2 \int_{R_1}^{R_2} \frac{\hat{u}_r^2}{r^2}\frac{\partial}{\partial r}(r^2 u_\theta^2)\,dr.$$

6

Note $\Phi \geq 0$, $k^2 \geq 0$ $\frac{\hat{u}_r^2}{r^2} \geq 0$.

Thus ~~If $\frac{\partial}{\partial r}(r^2 u_\theta^2) < 0$ throughout $R_1 < r < R_2$~~

2

~~then $\omega^2 < 0 \Rightarrow$ instability.~~

if $\omega^2 < 0$ (i.e. linearly unstable)

then $$\int_{R_1}^{R_2} \frac{\hat{u}_r^2}{r^2}\frac{\partial}{\partial r}(r^2 u_\theta^2)\,dr < 0$$

$\Rightarrow$ $\frac{\partial}{\partial r}(r^2 u_\theta^2) < 0$ for some r in $R_1 < r < R_2$

unseen, but similar to 2D example