1 (a)

$z = h$

$\underline{u} = u(z,t)\,\hat{\underline{x}}$

$z = 0$

Standard problem 4

(a) $\nabla \cdot \underline{u} = 0$ automatically.

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \qquad \text{(x component of Navier Stokes)}$$

Boundary conditions $\left.\begin{array}{l} u(0) = 0 \\ u(h) = 0 \end{array}\right\}$ no - slip

(b) Write $u = f(z)\, e^{2i\omega t}$ (and take real part)

Stokes layer in lectures. This is similar

$$2i\omega f = G + \nu f'' \quad \text{subject to } f(0) = 0 \quad f(h) = 0$$

$$f'' - \frac{2i\omega}{\nu} f = -\frac{G}{\nu} \qquad 2i = (1+i)^2, \quad \delta^2 = \frac{\omega}{\nu}$$

Try $f(z) = e^{mz}$: $f'' - \frac{2i\omega}{\nu} f = 0 \Rightarrow m^2 = \frac{2i\omega}{\nu} = (1+i)^2\delta^2$

General solution $f(z) = \frac{G}{2i\omega} + A e^{(1+i)\delta z} + B e^{-(1+i)\delta z}$

$f(0) = 0 \Rightarrow 0 = \frac{G}{2i\omega} + A + B$

$f(h) = 0 \Rightarrow 0 = \frac{G}{2i\omega} + A e^{(1+i)\delta h} + B e^{-(1+i)\delta h}$

$$\Rightarrow 0 = \frac{G}{2i\omega}\left(e^{(1+i)\delta h} - 1\right) + B\left(e^{(1+i)\delta h} - e^{-(1+i)\delta h}\right)$$

and $0 = \frac{G}{2i\omega}\left(e^{-(1+i)\delta h} - 1\right) + A\left(e^{-(1+i)\delta h} - e^{(1+i)\delta h}\right)$

$$\Rightarrow f(z) = \frac{G}{2i\omega}\left[1 + \frac{1 - e^{-(1+i)\delta h}}{e^{-(1+i)\delta h} - e^{(1+i)\delta h}} e^{(1+i)\delta z} + \frac{e^{(1+i)\delta h} - 1}{e^{-(1+i)\delta h} - e^{(1+i)\delta h}} e^{-(1+i)\delta z}\right]$$

$$= \frac{G}{2i\omega}\left[1 - \frac{\sinh[(1+i)\delta z]}{\sinh((1+i)\delta h)} + \frac{\sinh[(1+i)\delta(z-h)]}{\sinh((1+i)\delta h)}\right]$$

Hence $u(z,t) = \mathrm{Re}\left\{ \frac{G}{2i\omega} e^{2i\omega t}\left(1 - \frac{\sinh((1+i)\delta z)}{\sinh((1+i)\delta h)} + \frac{\sinh((1+i)\delta(z-h))}{\sinh((1+i)\delta h)}\right)\right\}$

12

(c) Average flow $\bar{u} = \frac{1}{h}\int_0^h u\, dz$

$$\int_0^h \sinh((1+i)\delta z)\, dz = \frac{1}{(1+i)\delta}\left(\cosh((1+i)\delta h) - 1\right)$$

$$\int_0^h \sinh((1+i)\delta(z-h))\, dz = \frac{1}{(1+i)\delta}\left(\cancel{\cos} 1 - \cosh((1+i)\delta h)\right)$$

Unseen but straightforward

So $\bar{u} = \mathrm{Re}\left\{ \frac{G}{2i\omega h} e^{2i\omega t}\left(h + \frac{2}{(1+i)\delta}\frac{(1 - \cosh((1+i)\delta h))}{\sinh((1+i)\delta h)}\right)\right\}$

3

(d) When $\delta h \ll 1$ $\quad \cosh(1+i)\delta h = 1 + \frac{((1+i)\delta h)^2}{2} + \frac{((1+i)\delta h)^4}{24}$

$\sinh(1+i)\delta h = (1+i)\delta h + \frac{((1+i)\delta h)^3}{6} + \dots$

$$\bar{u} = \mathrm{Re}\left\{ \frac{G}{2i\omega} e^{2i\omega t}\left(1 + \frac{2}{(1+i)\delta h}\frac{-\frac{((1+i)\delta h)^2}{2} - \frac{((1+i)\delta h)^4}{24} + \dots}{(1+i)\delta h\left(1 + \frac{(\delta h(1+i))^2}{6} + \dots\right)}\right)\right\}$$

$$= \mathrm{Re}\left\{ \frac{G}{2i\omega} e^{2i\omega t}\left(1 - \frac{\left(1 + \frac{1}{12}((1+i)\delta h)^2\right)}{1 + \frac{1}{6}((1+i)\delta h)^2}\right)\right\}$$

$$= \mathrm{Re}\left\{ \frac{G}{2i\omega} e^{2i\omega t}\left(1 - 1 + \frac{1}{12}(1+i)^2(\delta h)^2 + \dots\right)\right\}$$

$$= \mathrm{Re}\left\{ \frac{G}{\omega}\cdot\frac{1}{12}\cdot h^2\cdot\frac{\omega}{\nu}\, e^{2i\omega t}\right\} = \frac{1}{12}\frac{Gh^2}{\nu}\cos 2\omega t.$$

4

$h\delta \equiv \frac{\text{width of channel}}{\text{length of viscous diffusion}}$

4

Unseen

If $h\delta \ll 1$, then changes in velocity diffused rapidly across the channel

$\Rightarrow$ The flow adjust to pressure gradient more rapidly than the timescale of its variation.

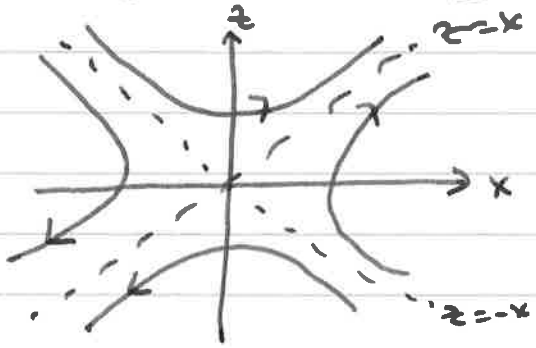
$\Rightarrow$ Flux $= G\cos 2\omega t \cdot \frac{1}{12}\frac{h^2}{\nu}$ where $\frac{1}{12}\frac{h^2}{\nu}$ is the 'steady response' for flow in a channel.

2

Answer in lectures

2(a) Flow field $u_i = E_{ij} x_j \Rightarrow \underline{u} = (z, 0, x)$

Streamlines given by $\dfrac{dx}{z} = \dfrac{dz}{x} \Rightarrow z^2 - x^2 = \text{constant}.$



(b) Given $u_i = E_{ij} x_j \left(1 - \frac{a^5}{r^5}\right) - \frac{5 x_k E_{kl} x_l}{2}\left(\frac{a^3}{r^5} - \frac{a^5}{r^7}\right) x_i$

Method in lectures

$$\nabla_j u_i = E_{ij}\left(1 - \frac{a^5}{r^5}\right) + \frac{5 E_{ik} x_k}{r^7} a^5 x_j \qquad \text{using } \nabla_j r = \frac{x_j}{r}$$

$$- \frac{5}{2} x_k E_{kl} x_l \left(\frac{a^3}{r^5} - \frac{a^5}{r^7}\right)\delta_{ij} - \frac{5 E_{jl} x_l}{2}\left(\frac{a^3}{r^5} - \frac{a^5}{r^7}\right) x_i$$

$$- \frac{5}{2} \cancel{E} x_k E_{kj}\left(\frac{a^3}{r^5} - \frac{a^5}{r^7}\right) x_i - \frac{5}{2} x_k E_{kl} x_l \left(-\frac{5a^3}{r^7} x_j + \frac{7a^5}{r^9} x_j\right) x_i$$

$$\nabla_j u_i\Big|_{r=a} = \frac{5 E_{ik} x_k}{a^2} x_j - \frac{5}{2} x_k E_{kl} x_l \cdot \frac{2}{a^4} x_i x_j$$

$$= \frac{5}{a^2} E_{ik} x_k x_j - \frac{5}{a^4} E_{kl} x_k x_l x_i x_j$$

5

Stress tensor $\sigma_{ij} = \mu\left(\nabla_j u_i + \nabla_i u_j\right) - p\,\delta_{ij}$

Hence $$\sigma_{ij}\Big|_{r=a} = \frac{5\mu}{a^2}\left(E_{ik} x_k x_j + E_{jk} x_k x_i\right) - \frac{10}{a^4} x_k E_{kl} x_l x_i x_j + \frac{5\mu}{a^2} x_k E_{kl} x_l \delta_{ij}$$

On surface of sphere $n_i = \frac{x_i}{a}$

$$\tau_i = \sigma_{ij} n_j = \frac{5\mu}{a^2}\left(E_{ik} x_k \cdot a + \frac{1}{a} x_j E_{jk} x_k x_i\right) - \frac{10}{a^3} x_k E_{kl} x_l x_i + \frac{5\mu}{a^2} x_k E_{kl} x_l \frac{x_i}{a} = \frac{5\mu}{a} E_{ik} x_k$$

5

unseen

(c) Drag on particle $= \int_{r=a} \tau_i \, dS = \frac{5\mu}{a} E_{ik} \int x_k \, dS = 0$

as $\int_{r=a} x_k \, dS = 0$

(d) Net torque on particle $G_i = \int_{r=a} \varepsilon_{ijk} x_j \tau_k \, dS$

unseen

$$= \int_{r=a} \varepsilon_{ijk} x_j \frac{5\mu}{a} E_{kp} x_p \, dS$$

$$= \frac{5\mu}{a} \varepsilon_{ijk} E_{kp} \int_{r=a} x_j x_p \, dS$$

$$= \frac{5\mu}{a} \varepsilon_{ijk} E_{kp} \cdot \frac{4\pi a^4}{3} \cdot \delta_{jp}$$

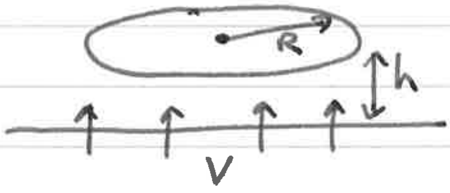
$$= \frac{20\pi a^{3}\mu}{3} \varepsilon_{ijk} E_{kj}$$

but $\varepsilon_{ijk} E_{kj} = - \varepsilon_{ikj} E_{kj} = - \varepsilon_{ikj} E_{jk}$

$= - \varepsilon_{ijk} E_{kj} = 0$

Hence net torque $\underline{G} = 0$.

3.



$h/R \ll 1$, $\quad \frac{\rho V h}{\mu} \ll 1$ } lubrication regime.

standard lubrication ~~boundary~~ layer method

(a) let velocity field $\underline{u} = u\,\hat{\underline{r}} + w\,\hat{\underline{z}}$

If $h/R \ll 1 \Rightarrow u \gg w$ Flow is predominantly horizontal

Then radial component of Navier-Stokes

$$\frac{\partial u}{\partial t} + \underline{u}\cdot\nabla u = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

- viscous term simplified due $h/R \ll 1$
- advective (inertial) terms $\sim \frac{u^2}{R}$, viscous terms $\sim \nu u/h^2$

Viscous dominates inertia if $\frac{\nu u}{h^2} \gg u^2/R$

$$\Rightarrow \quad 1 \gg \frac{\rho u h^2}{\mu R}$$

but $V/h \sim u/R \quad \Rightarrow \quad 1 \gg \frac{\rho V h}{\mu}$.

unseen example, but similar to lectures

(b) Inertia negligible $\quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\frac{\partial^2 u}{\partial z^2} \qquad u(0)=0,\ u(h)=0$

$$\Rightarrow \quad u = \frac{+1}{2\mu}\frac{\partial p}{\partial r}\left(z^2 - hz\right)$$

Flux $\quad q = \int_0^h u\,dz = \frac{1}{2\mu}\frac{\partial p}{\partial r}\left[\frac{z^3}{3} - \frac{hz^2}{2}\right]_0^h = \frac{-h^3}{12\mu}\frac{\partial p}{\partial r}$

Mass conservation $\quad \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$

$$\Rightarrow \quad \frac{1}{r}\int_0^h \frac{\partial}{\partial r}(ru)\,dz + w(h) - w(0) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial r}\int_0^h r u\,dz = rV$$

$$\Rightarrow \quad r q = \frac{1}{2} r^2 V$$

$$\Rightarrow \quad \frac{\partial p}{\partial r} = -\frac{6\mu r V}{h^3}$$

10

(c) $p = p_\infty + \frac{3\mu V}{h^3}(R^2 - r^2)$ using $p(R) = p_\infty$

Normal force $= 2\pi\int_0^R p\, r\, dr$ on both sides of dish

Excess force due to lubrication $= 2\pi\int_0^R (p-p_\infty)\, r\, dr$

$$= 2\pi \cdot \frac{3\mu V}{h^3}\left[\frac{1}{2}R^4 - \frac{1}{4}R^4\right]$$

$$= \frac{3\pi\mu V R^4}{2h^3}$$

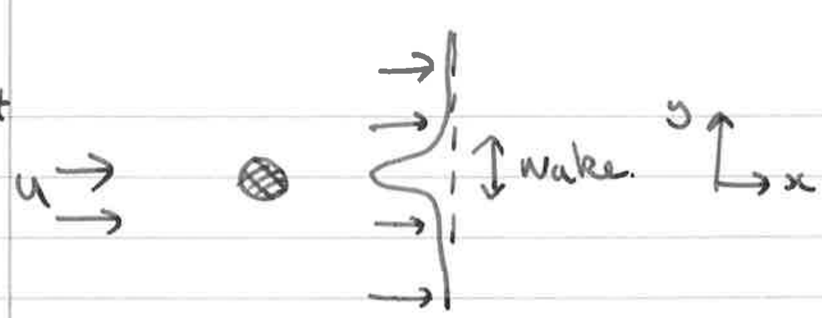
In equilibrium if $Mg = \frac{3\pi\mu V R^4}{2h^3}$

$$\Rightarrow \quad h^3 = \frac{3\pi\mu V R^4}{2Mg}$$

$$\Rightarrow \quad h = R\left(\frac{3\pi\mu V R}{2Mg}\right)^{1/3}$$

10

4



Standard

(a) Reynolds Number $= \dfrac{UD}{\nu}$

3 The wake is thin if $UD/\nu \gg 1$.

Unseen, but standard

(b) If $\underline{u} = (u, v)$ then steady momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(pressure uniform)

Now write $u = U - u_1$.

$$-U\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_1}{\partial x} + v\frac{\partial u_1}{\partial y} = -\nu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right)$$

Mass conservation $\quad -\dfrac{\partial u_1}{\partial x} + \dfrac{\partial v}{\partial y} = 0.$

Thus $\quad \left|U\dfrac{\partial u_1}{\partial x}\right| \gg \left|u_1\dfrac{\partial u_1}{\partial x}\right|, \left|v\dfrac{\partial u_1}{\partial y}\right|.$

Thin wake $\Rightarrow \quad \left|\dfrac{\partial^2 u_1}{\partial x^2}\right| \ll \left|\dfrac{\partial^2 u_1}{\partial y^2}\right|$

6 Hence to leading order $\quad U\dfrac{\partial u_1}{\partial x} = \nu\dfrac{\partial^2 u_1}{\partial y^2}.$

Boundary conditions: (i) $u_1 \to 0$ as $|y| \to \infty$ (decay as $|y| \to \infty$)
(ii) $\dfrac{\partial u_1}{\partial y} = 0$ at $y = 0$ (symmetry)

2

Straightforward

(c) $\displaystyle\int_{-\infty}^{\infty} U\frac{\partial u_1}{\partial x}\,dy = \left[\nu\frac{\partial u_1}{\partial y}\right]_{-\infty}^{\infty} = 0$

$\Rightarrow \quad U\dfrac{\partial}{\partial x}\displaystyle\int_{-\infty}^{\infty} u_1\,dy = 0 \quad \Rightarrow \int_{-\infty}^{\infty} u_1\,dy = Q$ (constant)

4

(d) Let $u_1 = \frac{Q}{x^{1/2}} f\left(\frac{y}{x^{1/2}}\right)$

$$\partial u_1 / \partial x = \frac{-Q}{2x^{3/2}} f - \frac{Q}{x^{1/2}} f' \cdot \frac{y}{2x^{3/2}}$$

$$\frac{\partial u_1}{\partial y} = \frac{Q}{x} f' \qquad \frac{\partial^2 u_1}{\partial y^2} = \frac{Q}{x^{3/2}} f''$$

Hence $$-\frac{U Q}{2x^{3/2}}\left(f + \frac{y}{x^{1/2}} f'\right) = \frac{\nu Q}{x^{3/2}} f''$$

$\eta = \frac{y}{x^{1/2}}$ $$-\frac{U}{2\nu}(f\eta)' = f''$$

$$\Rightarrow \quad -\frac{U}{2\nu}\eta f = f' + \text{const}$$

$\text{const} = 0$ using $\eta \to \infty$, $f \to 0$

$$\Rightarrow \quad \frac{f'}{f} = -\frac{U\eta}{2\nu}$$

$$\Rightarrow \quad f = c \cdot \exp\left(-\frac{U\eta^2}{4\nu}\right)$$

but $\int_{-\infty}^{\infty} u_1 \, dy = Q \Rightarrow cQ \int_{-\infty}^{\infty} \frac{1}{x^{1/2}} \cdot \exp\left(-\frac{Uy^2}{4\nu x}\right) dy = Q$

$Uy^2 = 4\nu x s^2 \Rightarrow y = 2\left(\frac{\nu x}{U}\right)^{1/2} s$

$$\Rightarrow \quad c \cdot 2\left(\frac{\nu}{U}\right)^{1/2} \int_{-\infty}^{\infty} e^{-s^2} ds = 1$$

But $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi} \Rightarrow c = \left(\frac{U}{4\pi\nu}\right)^{1/2}$

Hence $$u_1 = \left(\frac{UQ^2}{4\pi\nu x}\right)^{1/2} \exp\left(-\frac{Uy^2}{4\nu x}\right)$$

8

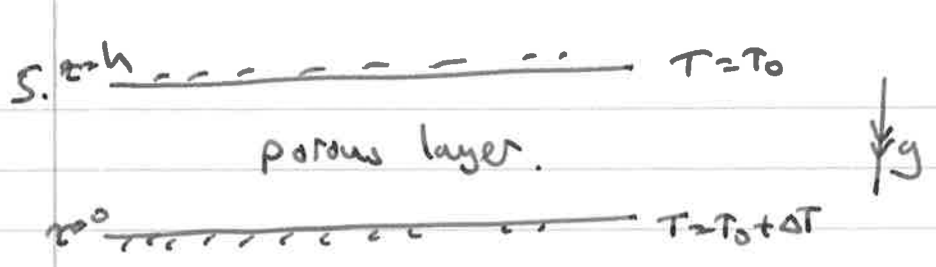
Thickness of boundary layer $y^2 \sim \frac{4\nu x}{U}$

$\Rightarrow$ thickness $y \sim \left(\frac{\nu x}{U}\right)^{1/2} \Rightarrow \frac{y}{D} \sim \frac{1}{Re}\left(\frac{x}{D}\right)^{1/2}$

2

Unseen

5.



on worksheet

(a) No flow, steady: $\nabla^2 T = 0$

$$0 = \frac{\partial p}{\partial z} + \rho g.$$

Temperature field $T = \overline{T}(z) = T_0 + \Delta T(1 - z/h)$

$$\rho = \overline{\rho}(z) = \rho_0\left(1 - \alpha\Delta T(1 - z/h)\right)$$

$$\frac{\partial \overline{p}}{\partial z} = -\overline{\rho} g = -\rho_0 g\left(1 - \alpha\Delta T(1 - z/h)\right)$$

$$\Rightarrow \quad \overline{p}(z) = -\rho_0 g\left(z - \alpha\Delta T\left(z - \frac{z^2}{2h}\right)\right) + \text{constant}.$$

3

Isotropic case on worksheet

(b) Introduce perturbation $T = \overline{T} + T'$ $\quad |T'/\overline{T}| \ll 1.$

$$\rho = \overline{\rho} + \rho' \quad |\rho'/\overline{\rho}| \ll 1$$

$$\rho = \rho_0\left(1 - \alpha(T - T_0)\right) \Rightarrow \rho' = -\alpha\rho_0 T' \quad \text{—— (1)}$$

$$\frac{\partial T}{\partial t} + \underline{u}\cdot\nabla T = \kappa\nabla^2 T \Rightarrow \left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)T' = -w\frac{d\overline{T}}{dz} = \frac{\Delta T}{h}w \quad \text{—— (2)}$$

$$u = -\frac{k_h}{\mu}\frac{\partial p'}{\partial x} \quad \text{—— (3)}$$

$$w = -\frac{k_v}{\mu}\left(\frac{\partial p'}{\partial z} + \rho' g\right) \quad \text{—— (4)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \text{—— (5)}$$

From —(3),(4),(5) $\quad -\frac{k_h}{\mu}\frac{\partial^2 p'}{\partial x^2} - \frac{k_v}{\mu}\frac{\partial^2 p'}{\partial z^2} - \frac{k_v}{\mu}g\frac{\partial \rho'}{\partial z} = 0$

$$\Rightarrow \quad -\frac{1}{\mu}\left(k_h\frac{\partial^2}{\partial x^2} + k_v\frac{\partial^2}{\partial z^2}\right)w = -\frac{k_v}{\mu}\left(\frac{\partial}{\partial z}\left(-\frac{k_h}{\mu}\frac{\partial^2 p'}{\partial x^2} - \frac{k_v}{\mu}\frac{\partial^2 p'}{\partial z^2}\right)\right.$$

$$\left. + g\left(-\frac{k_h}{\mu}\frac{\partial^2 \rho'}{\partial x^2} - \frac{k_v}{\mu}\frac{\partial^2 \rho'}{\partial z^2}\right)\right)$$

$$\Rightarrow \quad -\frac{1}{\mu}\left(K_h\frac{\partial^2}{\partial x^2} + K_v\frac{\partial^2}{\partial z^2}\right)w = -\frac{K_v}{\mu}\left(\frac{K_v}{\mu} g \frac{\partial^2 \rho'}{\partial z^2} - \frac{g}{\mu}\left(K_h\frac{\partial^2}{\partial x^2} + K_v\frac{\partial^2}{\partial z^2}\right)\rho'\right)$$

$$= \frac{g K_v K_h}{\mu^2}\frac{\partial^2 \rho'}{\partial x^2}$$

$$= -\frac{\alpha \rho_0 g K_v K_h}{\mu^2}\frac{\partial^2 T'}{\partial x^2}$$

Then use –(2) to find

$$\frac{1}{\mu}\left(K_h\frac{\partial^2}{\partial x^2} + K_v\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial}{\partial t} - \mathcal{K}\nabla^2\right)w = \frac{\alpha \Delta T \rho_0 g K_v K_h}{\mu^2 h}\frac{\partial^2 w}{\partial x^2}$$

10

(c) Now substitute $w = e^{ikx}\sin(\pi n z/h)$

$$\Rightarrow \quad \left(-K_h k^2 - \left(\frac{n\pi}{h}\right)^2 K_v\right)\mathcal{K}\left(k^2 + \left(\frac{n\pi}{h}\right)^2\right) = -k^2\frac{\alpha \Delta T \rho_0 g K_v K_h}{\mu h}$$

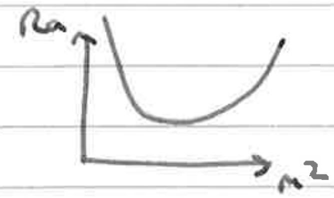
$$\Rightarrow \quad \frac{\alpha \Delta T \rho_0 g K_v h}{\mu \mathcal{K}} = \frac{\left((kh)^2 + n^2\pi^2\right)\cdot\left((kh)^2 + \frac{K_v}{K_h}n^2\pi^2\right)}{(kh)^2}$$

Define $\quad Ra = \dfrac{\alpha \Delta T \rho_0 g K_v h}{\mu \mathcal{K}}$

standard method

7

(d) $\quad Ra = \dfrac{(m^2 + n^2\pi^2)\left(m^2 + \frac{K_v}{K_h}n^2\pi^2\right)}{m^2} \qquad m = kh$



$$\frac{\partial Ra}{\partial m^2} = -\frac{1}{m^4}(m^2 + n^2\pi^2)\left(m^2 + \frac{K_v}{K_h}n^2\pi^2\right) + \left(2m^2 + n^2\pi^2(1 + K_v/K_h)\right)\frac{1}{m^2}$$

$$= 0 \quad \text{at} \quad -m^4 - m^2 n^2\pi^2\left(1 + \frac{K_v}{K_h}\right) - \frac{K_v}{K_h}(n^2\pi^2)^2 + 2m^4 + m^2 n^2\pi^2(1 + K_v/K_h) = 0$$

$$\Rightarrow \quad m^4 = \frac{K_v}{K_h}(n^2\pi^2)^2 \quad \Rightarrow \quad Ra_{min} = \frac{1}{(K_v/K_h)^{1/2}}\frac{1}{n^2\pi^2}\left(\left(\frac{K_v}{K_h}\right)^{1/2} + 1\right)n^2\pi^2\left(\left(\frac{K_v}{K_h}\right)^{1/2} + \frac{K_v}{K_h}\right)n^2\pi^2$$

$$= n^2\pi^2\left(1 + \left(\frac{K_v}{K_h}\right)^{1/2}\right)^2$$

5 and minimum occurs at $n=1 \Rightarrow Ra = \pi^2\left(1 + \left(\frac{K_v}{K_h}\right)^{1/2}\right)^2$

Under but isotropic result covered.