Standard problem

1. (a) $\underline{u} = u(y)\,\hat{\underline{x}}$ satisfies $\nabla\cdot\underline{u} = 0$ automatically

$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u}$ simplifies to $0 = G + \mu \frac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \qquad \frac{\partial^2 u}{\partial y^2} = \begin{cases} -\dfrac{G}{\mu_1} & -a < y < 0 \\ -\dfrac{G}{\mu_2} & 0 < y < a \end{cases}$$

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Boundary conditions : (i) No slip at $y = a$ $\quad u(a) = 0$

(ii) No slip at $y = -a$ $\quad u(-a) = 0$

(iii) Continuity of velocity at $y = 0$ $\quad u(0^-) = u(0^+)$

(iv) Continuity of stress at $y = 0$ $\quad \mu_1 \frac{du}{dy}(0^-) = \mu_2 \frac{du}{dy}(0^+)$

(b)
$$u = \begin{cases} -\dfrac{G}{2\mu_1} y^2 + Ay + B & -a < y < 0 \\ -\dfrac{G}{2\mu_2} y^2 + Cy + D & 0 < y < a \end{cases}$$

Straightforward example

Apply continuity conditions at $y = 0$ $\quad B = D$ —(1)

$\mu_1 A = \mu_2 C$ —(2)

No slip at $y = -a$ $\quad -\frac{G}{2\mu_1} a^2 - Aa + B = 0$ —(3)

No slip at $y = +a$ $\quad -\frac{G}{2\mu_2} a^2 + Ca + D = 0$ —(4)

$\Rightarrow \quad +\frac{Ga^2}{2}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) + a(A + C) = 0 \qquad C = \frac{\mu_1}{\mu_2} A$

$\Rightarrow \quad +\frac{Ga^2}{2} \frac{(\mu_2 - \mu_1)}{\mu_1 \mu_2} + aA \frac{(\mu_1 + \mu_2)}{\mu_2} = 0$

$\Rightarrow \quad A = -\frac{Ga}{2} \frac{(\mu_2 - \mu_1)}{\mu_1(\mu_1 + \mu_2)}$ and $C = -\frac{Ga}{2} \frac{(\mu_2 - \mu_1)}{\mu_2(\mu_1 + \mu_2)}$

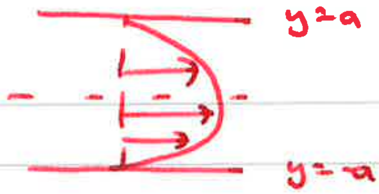
Into —(3) $\quad B = \frac{Ga^2}{2\mu_1} - \frac{Ga^2}{2} \frac{(\mu_2 - \mu_1)}{\mu_1(\mu_1 + \mu_2)} = \frac{Ga^2}{2\mu_1} \cdot \frac{2\mu_1}{\mu_1 + \mu_2}$

Hence $\quad u = -\frac{Ga^2}{2\mu_1}\left[\frac{y^2}{a^2} - \frac{(\mu_1 - \mu_2)}{\mu_1 + \mu_2}\frac{y}{a} - \frac{2\mu_1}{\mu_1 + \mu_2}\right] \qquad -a < y < 0$

5

$\quad = -\frac{Ga^2}{2\mu_2}\left[\frac{y^2}{a^2} - \frac{(\mu_1 - \mu_2)}{\mu_1 + \mu_2}\frac{y}{a} - \frac{2\mu_2}{\mu_1 + \mu_2}\right] \qquad 0 < y < a$

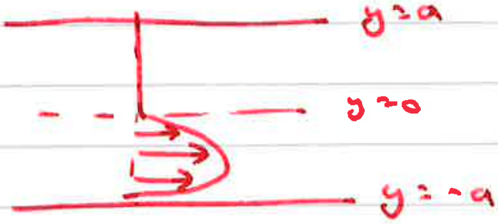
(b) (i) When $\mu_1=\mu_2=\mu$ $u = -\frac{Ga^2}{2\mu}\left[\frac{y^2}{a^2}-1\right]$



(ii) When $\mu_1/\mu_2 \ll 1$ $u \simeq \begin{cases} -\frac{Ga^2}{2\mu_1}\left(\frac{y^2}{a^2}+\frac{y}{a}\right) & -a<y<0 \\ 0 & 0<y<a \end{cases}$

unseen

2



(c) Shear stress $\underline{\tau}_1 = \mu_1 \frac{du}{dy}\Big|_{y=-a}\hat{\underline{x}} = -\frac{Ga^2}{2}\left(\frac{2y}{a^2} - \frac{(\mu_1-\mu_2)}{\mu_1+\mu_2}\frac{1}{a}\right)\Big|_{y=-a}\hat{\underline{x}}$

$$= -\frac{Ga}{2}\left(-2-\frac{(\mu_1-\mu_2)}{\mu_1+\mu_2}\right) = \frac{Ga(3\mu_1+\mu_2)}{2(\mu_1+\mu_2)}\hat{\underline{x}}$$

Shear stress $\underline{\tau}_2 = -\mu_2\frac{du}{dy}\Big|_{y=a}\hat{\underline{x}} = +\frac{Ga^2}{2}\left(\frac{2y}{a} - \frac{(\mu_1-\mu_2)}{\mu_1+\mu_2}\frac{1}{a}\right)\Big|_{y=a}\hat{\underline{x}}$

$$= +\frac{Ga}{2}\frac{(\mu_1+3\mu_2)}{(\mu_1+\mu_2)}\hat{\underline{x}}$$

standard, but unseen

4

$$\underline{\tau}_1+\underline{\tau}_2 = \frac{Ga}{2}\frac{(4\mu_1+4\mu_2)}{\mu_1+\mu_2}\hat{\underline{x}} = 2Ga\,\hat{\underline{x}}$$

In the Navier Stokes equation $0 = G + \frac{d\sigma_{xy}}{dy}$ σ_{xy} = deviatoric shear stress

$$\Rightarrow \quad 0 = 2aG + \sigma_{xy}\big|_{y=a} - \sigma_{xy}\big|_{y=-a}$$

2 $\Rightarrow \quad \tau_1+\tau_2 = 2aG.$

(d) Volume flux $q_1 = \int_{-a}^{0} u\,dy = -\frac{Ga^2}{2\mu_1}\left[\frac{y^3}{3a^2} - \frac{(\mu_1-\mu_2)y^2}{(\mu_1+\mu_2)2a} - \frac{2\mu_1 y}{\mu_1+\mu_2}\right]_{-a}^{0}$

unseen, but standard methods

$$= -\frac{Ga^3}{2\mu_1}\left(\frac{1}{3} + \frac{(\mu_1-\mu_2)}{2(\mu_1+\mu_2)} - \frac{2\mu_1}{\mu_1+\mu_2}\right)$$

$$= -\frac{Ga^3}{12\mu_1}\frac{\left(2(\mu_1+\mu_2)+3(\mu_1-\mu_2)-12\mu_1\right)}{\mu_1+\mu_2}$$

$$= \frac{Ga^3}{12\mu_1}\frac{(7\mu_1+\mu_2)}{\mu_1+\mu_2}$$

$$q_2 = \int_0^a u\,dy = \frac{Ga^3}{12\mu_2}\frac{(7\mu_2+\mu_1)}{(\mu_1+\mu_2)} \qquad \text{by symmetry}$$

Hence $q_1 + q_2 = \dfrac{Ga^3}{12(\mu_1+\mu_2)}\left(7 + \dfrac{\mu_2}{\mu_1} + 7 + \dfrac{\mu_1}{\mu_2}\right)$

4

(i) When $\mu_1 = \mu_2 = \mu$. $\qquad q_1+q_2 = \dfrac{Ga^3}{24\mu}.16 = \dfrac{G(2a)^3}{12\mu}.$

(ii) When $\mu_1/\mu_2 \ll 1$ $\qquad q_1+q_2 \simeq \dfrac{Ga^3}{12.\mu_2}.\dfrac{\mu_2}{\mu_1} = \dfrac{Ga^3}{12\mu_1}$

4

2 (a) Force on sphere $f_i = \int_S \sigma_{ij} n_j \, dS = \int_{r=a} \frac{3\mu}{2a} u_i \cdot a^2 \sin\theta \, d\theta \, d\phi$

in notes 4

$= 6\pi\mu a u_i$

(b) $\sigma_{ij} = -p\delta_{ij} + \mu(\nabla_i u_j + \nabla_j u_i)$ $\quad \nabla_j \sigma_{ij} = 0$ (Stokes equation)

$\nabla_i u_i = 0$ (Incompressible)

$$\nabla_j (\sigma_{ij}^{(1)} u_i^{(2)} - \sigma_{ij}^{(2)} u_i^{(1)}) = u_i^{(2)} \nabla_j \sigma_{ij}^{(1)} + \sigma_{ij}^{(1)} \nabla_j u_i^{(2)} - u_i^{(1)} \nabla_j \sigma_{ij}^{(2)} - \sigma_{ij}^{(2)} \nabla_j u_i^{(1)}$$ using $\nabla \cdot \underline{\underline{\sigma}} = 0$

$$= (-p^{(1)}\delta_{ij} + \mu(\nabla_j u_i^{(1)} + \nabla_i u_j^{(1)}))\nabla_j u_i^{(2)} - (-p^{(2)}\delta_{ij} + \mu(\nabla_j u_i^{(2)} + \nabla_i u_j^{(2)})\nabla_j u_i^{(1)}$$ using $\nabla \cdot \underline{u} = 0$

$$= \mu(\nabla_i u_j^{(1)} \nabla_j u_i^{(2)} - \nabla_i u_j^{(2)} \nabla_j u_i^{(1)})$$

$$= 0$$

Hence $\int_V \nabla_j (\sigma_{ij}^{(1)} u_i^{(2)} - \sigma_{ij}^{(2)} u_i^{(1)}) \, dV = 0$

Answer

10 $\quad \Rightarrow \int_S \sigma_{ij}^{(1)} n_j u_i^{(2)} - \sigma_{ij}^{(2)} n_j u_i^{(1)} \, dS = 0$

(c) let flow field $\underline{u}$ around sphere satisfies $\mu \nabla^2 \underline{u} = \nabla p$

$\underline{u} \to \underline{u}^*$ as $r \to \infty$

$\int_{r=a} \sigma_{ij} n_j \, dS = 0$ (force free)

Velocity at centre of sphere is denoted $\underline{u}_c$

Define $\underline{v} = \underline{u} - \underline{u}^*$ so $\mu \nabla^2 \underline{v} = \nabla \hat{p}$, $\underline{v} \to 0$ as $r \to \infty$

$$\int_{r=a} \sigma_{ij}(v) n_j \, dS = \int_{r=a} (\sigma_{ij}(u) - \sigma_{ij}(u_*)) n_j \, dS$$

$= 0$ as $\nabla_j \sigma_{ij}(u_*) = 0$ in $r<a$.

Then treat field 1 : $\left.\begin{array}{l}\sigma_{ij}^{(1)} n_j = \dfrac{3\mu U_i}{2a} \\ u_i^{(1)} = U_i\end{array}\right\}$ on $r=a$

and field 2 : $\left.\begin{array}{l}\sigma_{ij}^{(2)} n_j \equiv \left(\sigma_{ij}(u) - \sigma_{ij}(u^*)\right) n_j \\ v_i = u_i^{(2)} = u_{c_i} - u_{*_i}\end{array}\right\}$ on $r=a$

Also $|v|$ and $|\sigma_{ij}(v)| \to 0$ as $r \to \infty$

Hence $\displaystyle\int_{r=a} \frac{3\mu U_i}{2\mu} . (\overset{u_{c_i} - u_{*_i}}{\cancel{\sigma_{ij}(u) - \sigma_{ij}(u^*)}}) n_j \, dS = \int_{r=a} U_i \left(\sigma_{ij}(u) - \sigma_{ij}(u^*)\right) n_j \, dS$

$\displaystyle = U_i \int_{r=a} \left(\sigma_{ij}(u) - \sigma_{ij}(u^*)\right) n_j \, dS$

$= 0$

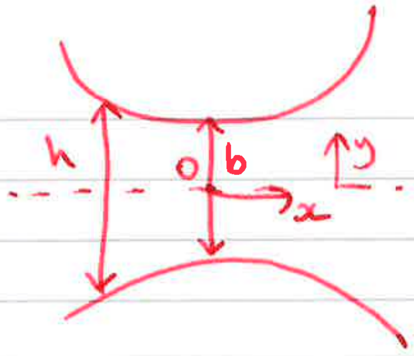
Unseen

$\Rightarrow \quad \displaystyle 4\pi a^2 u_{c_i} = \int_{r=a} u_{*_i} \, dS$ as U_i arbitrary

$\Rightarrow \quad \displaystyle \underline{u}_c = \frac{1}{4\pi a^2} \int_{r=a} \underline{u}^*(x) \, dS.$

11

3.



similar to example

(a) $\left(a + \frac{b}{2} - \frac{h}{2}\right)^2 + x^2 = a^2$

$\Rightarrow \left(a + \frac{b}{2} - \frac{h}{2}\right) = a\left(1 - \frac{x^2}{a^2}\right)^{1/2}$

$\Rightarrow \frac{h}{2} = a + \frac{b}{2} - a\left(1 - \frac{x^2}{a^2}\right)^{1/2}$

$= a + \frac{b}{2} - a\left(1 - \frac{1}{2}\frac{x^2}{a^2} + \dots\right)$

$= \frac{b}{2} + \frac{x^2}{2a}$

4 $\Rightarrow h = b + \frac{x^2}{a}.$

(b) $\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$ $\qquad u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + Ay + B$

similar to example

Boundary conditions $u\left(\pm \frac{h}{2}\right) = 0 \Rightarrow 0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{h^2}{4} - A\frac{h}{2} + B$

and $0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{h^2}{4} + A\frac{h}{2} + B$

$\Rightarrow A = 0, \quad B = -\frac{1}{2\mu}\frac{h^2}{4}\frac{\partial p}{\partial x} \Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2/4\right)$

Volume flux per unit width $q = \int_{-h/2}^{h/2} u\,dy = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{y^3}{3} - \frac{h^2 y}{4}\right]_{-h/2}^{h/2}$

$= \frac{1}{\mu}\frac{\partial p}{\partial x}\left(\frac{h^3}{24} - \frac{h^3}{8}\right) = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}$

(c) q = constant $\Rightarrow \frac{\partial p}{\partial x} = -\frac{12\mu q}{h^3}$

$\int_{-\infty}^{\infty} \frac{\partial p}{\partial x}\,dx = -12\mu q \int_{-\infty}^{\infty} \frac{1}{(b + x^2/a)^3}\,dx$

$x^2/a = b\tan^2\theta \rightarrow x = \sqrt{ab}\tan\theta \qquad dx/d\theta = \sqrt{ab}\sec^2\theta$

$p(\infty) - p(-\infty) = -\Delta p = -12\mu q\sqrt{ab}\,\frac{1}{b^3}\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta}{\sec^6\theta}\,d\theta.$

$\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \int_{-\pi/2}^{\pi/2} (\cos 2\theta + 1)^2 \frac{1}{4}\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\,d\theta$

$= \int_{-\pi/2}^{\pi/2} \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\,d\theta = \frac{3}{8}\pi$

unseen

Here $\Delta p = \frac{12\mu q\sqrt{ab}}{b^3}\,\frac{3}{8}\pi$

$$\Rightarrow \quad q = \frac{2\Delta p\, b^3}{9\pi\mu\sqrt{ab}}$$

(d) Cylinders rotate so that $u = a\Omega$ at $y = \frac{h}{2}$

unseen

Thus $u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - \frac{h^2}{4}\right) + \Omega a$

$$q = \frac{-h^3}{12\mu}\frac{\partial p}{\partial x} + \Omega a h$$

$$\Rightarrow \quad \frac{\partial p}{\partial x} = -12\mu\frac{(q - \Omega a h)}{h^3}$$

Need $\int_{-\infty}^{\infty}\frac{\Omega a h}{h^3}\,dx = \frac{\Omega a}{b^2}\int_{-\infty}^{\infty}\left(1+\frac{x^2}{ab}\right)^{-2}dx$ $\qquad x = \sqrt{ab}\tan\theta$

$$= \frac{\Omega a}{b^2}\cdot\sqrt{ab}\int_{-\pi/2}^{\pi/2}(1+\tan^2\theta)^{-2}\sec^2\theta\,d\theta$$

$$= \frac{\Omega a\sqrt{ab}}{b^2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$\int\cos^2\theta\,d\theta = \int \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta$

$$= \frac{\Omega a\sqrt{ab}}{b^2}\cdot\frac{\pi}{2}$$

Thus $\Delta p = 12\mu q\cdot\frac{\sqrt{ab}}{b^3}\frac{3}{8}\pi - 12\mu\Omega a\frac{\sqrt{ab}}{b^2}\frac{\pi}{2}$

$$\Rightarrow \quad \Delta p = \frac{12\mu\sqrt{ab}}{b^3}\frac{3\pi}{8}\left(q - \Omega ab\frac{4}{3}\right)$$

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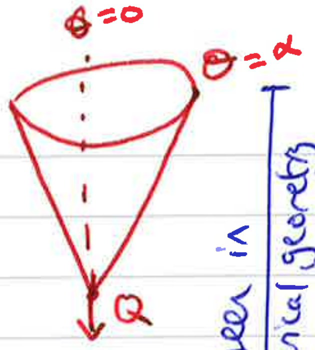
$$\Rightarrow \quad q = \frac{2\Delta p\, b^3}{9\pi\mu\sqrt{ab}} + \frac{4}{3}\Omega ab$$

4.



unseen in spherical geometry

(a) Volume flux across surface $r=r_0$ $= \displaystyle\int_0^{2\pi}\int_0^{\alpha} -\frac{A}{r^2}\cdot r^2\sin\theta\, d\theta\, d\alpha$

$$= \left[+A.2\pi\cos\theta\right]_0^{\alpha}$$

$$\Rightarrow \quad -Q = 2\pi A(\cos\alpha - 1)$$

6

(b) Reynolds number: $= \text{``}\frac{uL}{\nu}\text{''} = \frac{A}{r\nu} = \frac{Q}{2\pi(1-\cos\alpha)r\nu}$

unseen

Note that $Re \gg 1$ when $r \ll Q/\nu$

3

(c)

Bookwork

Boundary layer equations: $\underline{u} = (u, v)$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}$$

$$0 = -\frac{\partial p}{\partial y}$$

3

(d) Balance viscous stresses $\nu\, u/\delta^2$ with inertia u^2/x

Standard technique

$$\Rightarrow \quad \delta^2 \sim \frac{\nu x}{u} \qquad \text{but } u \sim \frac{A}{x^2}$$

$$\Rightarrow \quad \delta^2 \sim \frac{\nu x^3}{A}$$

Define $\delta(x) = \left(\frac{\nu x^3}{A}\right)^{1/2}$ and streamfunction $\psi = \frac{A}{x^2}\cdot\left(\frac{\nu x^3}{A}\right)^{1/2} F\left(\frac{y}{\delta}\right)$

$$= \left(\frac{A\nu}{x}\right)^{1/2} F\left(\frac{y}{\delta}\right)$$

6

(e) $u = \frac{\partial\psi}{\partial y} = \left(\frac{A\nu}{x}\right)^{1/2}\frac{1}{\delta}F' \qquad v = -\frac{\partial\psi}{\partial x} = +\frac{1}{2}\left(\frac{A\nu}{x^3}\right)^{1/2}F + \left(\frac{A\nu}{x}\right)^{1/2}\frac{y}{\delta}\cdot\frac{3}{2x}F'$

$$= \frac{A}{x^2}F'$$

$$\frac{\partial u}{\partial x} = \frac{-2A}{x^3}F' - \frac{3}{2x}\frac{A}{x^2}\frac{y}{\delta}F'' \qquad \frac{\partial u}{\partial y} = \frac{A}{x^2}\frac{1}{\delta}F'' \qquad \frac{\partial^2 u}{\partial y^2} = \frac{A}{x^2}\frac{1}{\delta^2}F'''$$

Unseen, but similar method in lectures

Pressure field $u\frac{du}{dx} = -\frac{1}{\rho}\frac{dp}{dx}$ outside of boundary layer

$$\Rightarrow \quad -\frac{1}{\rho}\frac{dp}{dx} = -\frac{A}{x^2}\cdot\frac{+2A}{x^3} = \frac{-2A^2}{x^5} \quad \text{as } u = -\frac{A}{x^2}$$

Writing $\zeta = y/\delta(x)$

Boundary layer equation:

$$\frac{A}{x^2}F'\left(\frac{-2A}{x^3}F' - \frac{3A}{x^3}\zeta F''\right) + \left(+\frac{1}{2}\left(\frac{A\nu}{x^3}\right)^{1/2}F + \left(\frac{A\nu}{x}\right)^{1/2}\frac{3}{2x}\zeta F'\right)\cdot\frac{A}{x^2\delta}F''$$

$$= -\frac{2A^2}{x^5} + \frac{\nu A}{x^2}\frac{1}{\delta^2}F'''$$

Substitute for $\delta^2 = \nu x^3/A$

$$-2F'^2 + \frac{1}{2}FF'' = -2 + F'''$$

Boundary conditions:

(i) $u(0) = 0 \Rightarrow F'(0) = 0$ No slip

(ii) $v(0) = 0 \Rightarrow F(0) = 0$ No normal flow

(iii) $u \to -\frac{A}{x^2}$ as $y \to \infty \Rightarrow F' \to -1$ Match outer flow field.

5 (a) Incompressible $\qquad \nabla\cdot\underline{u} = 0 \xrightarrow{\text{linearise}} \partial_x u_1 + \partial_z w_1 = 0$

Euler $\qquad \dfrac{D\underline{u}}{Dt} = -\dfrac{1}{\rho}\nabla p \qquad$
$$(\partial_t + u_0\partial_x)u_1 + w_1\partial_z u_0 = -\frac{1}{\rho}\partial_x p$$
$$(\partial_t + u_0\partial_x)w_1 = -\frac{1}{\rho}\partial_z p$$

So $$-\frac{1}{\rho}\nabla^2 p = 2\,\partial_x w_1\,\partial_z u_0 .$$

Then $$-\frac{1}{\rho}\partial_z\nabla^2 p = \nabla^2(\partial_t + u_0\partial_x)w_1$$
$$= (\partial_t + u_0\partial_x)\nabla^2 w_1 + 2\,\partial_z u_0\,\partial_x\partial_z w_1 + \partial_z\partial_z u_0\,\partial_x w_1$$

$$\Rightarrow \quad (\partial_t + u_0\partial_x)\nabla^2 w_1 - \partial_z\partial_z u_0\,\partial_x w_1 = 0$$

Now write $\quad w_1 = \hat{w}(z)\,e^{ik(x-ct)}$

$$ik(u_0 - c)\left(\frac{d^2\hat{w}}{dz^2} - k^2\hat{w}\right) - u_0'' ik\hat{w} = 0$$

6 $$\Rightarrow \quad \frac{d^2\hat{w}}{dz^2} - k^2\hat{w} - \frac{u_0''}{u_0 - c}\hat{w} = 0$$

Similar to lectures

(b) $$u_0 = \begin{cases} 1 & z > 1 \\ z & |z| \le 1 \\ -1 & z < -1 . \end{cases}$$

(i) Continuity of normal velocity at $z = \pm 1$

$\Rightarrow \quad \hat{w}$ continuous at $z = \pm 1$

2

(ii) Continuity of pressure at $z = \pm 1$

$\Rightarrow \quad p$ continuous at $z = \pm 1$

$\Rightarrow \quad \partial_x p$ continuous at $z = \pm 1$

4 $\Rightarrow \quad ik(u_0 - c)\hat{u} + u_0'\hat{w}$ continuous at $z = \pm 1$

But $ik\hat{u} + \dfrac{d\hat{w}}{dz} = 0 \quad \Rightarrow \quad u_0'\hat{w} - (u_0 - c)\dfrac{d\hat{w}}{dz}$ continuous at $z = \pm 1$

Similar to worksheets

(c) Write solution $\hat{w} = \begin{cases} Ae^{-kz} & z>1 \\ e^{-kz} + Be^{kz} & |z|\le 1 \\ De^{kz} & z<-1 \end{cases}$ [satisfies $\frac{d^2\hat{w}}{dz^2} - k^2\hat{w} = 0$ with $|\hat{w}| \to 0$ as $z \to \pm\infty$]

At $z=1$
$$Ae^{-k} = e^{-k} + Be^{k} \quad -(1)$$
$$-(1-c)\cdot -kAe^{-k} = e^{-k} + Be^{k} - (1-c)(-ke^{-k} + kBe^{k}) \quad -(2)$$

At $z=-1$
$$De^{-k} = e^{k} + Be^{-k} \quad -(3)$$
$$-(-1-c)kDe^{-k} = e^{k} + Be^{-k} - (-1-c)(-ke^{k} + kBe^{-k}) \quad -(4)$$

From $-(1)$ and $-(2)$ $k(1-c)(e^{-k} + Be^{k}) = e^{-k} + Be^{k} - (1-c)k(-e^{-k} + Be^{k})$

$$\Rightarrow k(1-c) - k(1-c) - 1 = Be^{2k}(1 - 2k(1-c))$$
$$\Rightarrow Be^{2k} = \frac{1}{2k(1-c) - 1}$$

From $-(3)$ and $-(4)$ $(1+c)k(e^{k} + Be^{-k}) = e^{k} + Be^{-k} + (1+c)k(-e^{k} + Be^{-k})$

$$\Rightarrow 2k(1+c) - 1 = Be^{-2k}(1 + 0)$$

Hence
$$e^{4k} = \frac{1}{(2k(1-c)-1)}\,\frac{1}{2k(1+c)-1}$$
$$\rightarrow e^{-4k} = (1-2k+2kc)(1-2k-2kc)$$
$$= (1-2k)^2 - 4k^2c^2$$

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9 $\Rightarrow$
$$c^2 = \frac{(1-2k)^2 - e^{-4k}}{4k^2}.$$

Instability? $c = c_r + ic_I$ $\quad |e^{-ikct}| = e^{kc_I t}$ requires $c_I > 0$.
$\Rightarrow c^2 < 0.$

For instability $(1-2k)^2 - e^{-4k} < 0$
$$\left((1-2k) - e^{-2k}\right)(1-2k+e^{-2k}) < 0$$

but $1-2k-e^{-2k} < 0$ for $k>0$ $\Rightarrow$ $1-2k+e^{-2k} > 0$ for instability

4 $\rightarrow$ For instability $0 < k < k_*$
where $1 - 2k_* + e^{-2k_*} = 0$