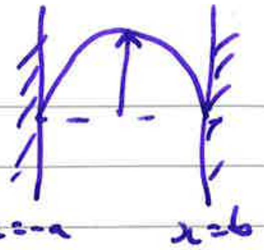
1(a) Seek fully developed flow $\underline{u} = w(x)\hat{\underline{z}}$



$\nabla \cdot \underline{u} = 0$ (automatically)

$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} + \rho \underline{g}$ reduces to $\mu \frac{d^2 w}{dz^2} = -G + \rho_1 g$

Integrate subject to no slip at $x=-a$ and $x=b$. $w(-a) = w(b) = 0$.

$$w = \frac{(\rho_1 g - G)}{2\mu}(x^2 + Ax + B) \quad A \text{ and } B \text{ to be determined}$$

$w(-a) = 0 \Rightarrow a^2 - Aa + B = 0 \Rightarrow a^2 - b^2 = A(a+b)$

$w(b) = 0 \Rightarrow b^2 + Ab + B = 0 \Rightarrow A = a-b, \; B = -ab$

Hence $w = \frac{\rho_1 g - G}{2\mu}\left(x^2 + (a-b)x - ab\right) = \frac{\rho_1 g - G}{2\mu}(x+a)(x-b)$

Volume flux per unit width $= \int_{-a}^{b} w \, dx = \int_0^{a+b} w(s) \, ds \quad (s = x+a)$

$$= \int_0^{a+b} \frac{\rho_1 g - G}{2\mu} s(s-a-b) \, ds$$

$$= \frac{(G - \rho_1 g)}{12\mu}(a+b)^3$$

Standard calculation [Result in lectures]

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(b)

$-a < x < 0$: $\mu \frac{d^2 w_1}{dx^2} = -G + \rho_1 g = \frac{(\rho_1 - \rho_2)g}{2} - \lambda\mu$

$0 < x < b$: $\mu \frac{d^2 w_2}{dx^2} = -G + \rho_2 g = \frac{(\rho_2 - \rho_1)g}{2} - \lambda\mu$

Define $m = \frac{(\rho_2 - \rho_1)g}{2\mu}$

So, $\frac{d^2 w_1}{dx^2} = -m - \lambda$ and $\frac{d^2 w_2}{dx^2} = m - \lambda$

Integrating
$$\left.\begin{aligned} w_1 &= -(m+\lambda)\frac{x^2}{2} + Ax + B \\ w_2 &= (m-\lambda)\frac{x^2}{2} + Cx + D \end{aligned}\right\} \; A, B, C, D \text{ to be determined.}$$

Unseen, though two fluid problems in examples

No slip $w_1(-a)=0 \doteq -(m+\lambda)\frac{a^2}{2} - Aa + B$

$w_2(b)=0 = (m-\lambda)b^2/2 + Cb + D$

Interfacial conditions : $[w]_{x=0^-}^{x=0^+}=0 \Rightarrow B=D$

$\left[\frac{dw}{dx}\right]_{x=0^-}^{x=0^+}=0 \Rightarrow A=C$

Then $-\frac{1}{2}(m+\lambda)a^2 - Aa - \frac{1}{2}(m-\lambda)b^2 - Cb = 0$

$$\Rightarrow A=C=\frac{\lambda(b^2-a^2)-m(a^2+b^2)}{2(a+b)}$$

and $D=B=(m+\lambda)\frac{a^2}{2} + a\cdot\frac{(\lambda(b^2-a^2)-m(a^2+b^2))}{2(a+b)}$

$$=\frac{\lambda(a^2b+ab^2)+m(a^2b-ab^2)}{2(a+b)}$$

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$$=\frac{ab}{2(a+b)}\left(\lambda(a+b)+m(a-b)\right)$$

(c) Total flux $=\int_{-a}^{0} w_1\,dx + \int_0^b w_2\,dx$

$$=\int_0^b \frac{mx^2}{2}dx - \int_{-a}^0 mx^2\,dx - \cancel{\lambda}\int_{-a}^b \frac{\lambda x^2}{2} + Ax + B\,dx$$

$$= \frac{m}{6}(b^3-a^3) - \frac{\lambda}{6}(b^3+a^3) + \frac{A}{2}(b^2-a^2) + B(a+b)$$

$$= \frac{m}{6}(b^3-a^3) - \frac{\lambda}{6}(b^3+a^3) + \frac{(b-a)}{4}\left(\lambda(b^2-a^2)-m(a^2+b^2)\right) + \frac{ab}{2}\left(\lambda(a+b) + m(a-b)\right)$$

$$= \frac{m}{12}\left(2b^3-2a^3-3(b-a)(a^2+b^2)+6ab(a-b)\right) - \frac{\lambda}{12}\left(2b^3+2a^3 \mp 3(b-a)(b^2-a^2) - 6ab(a+b)\right)$$

Unseen

$$= \frac{m}{12}\left(a^3-b^3+3a^2b-3ab^2\right) + \frac{\lambda}{12}\left(b^3+a^3+3a^2b+3ab^2\right)$$

Flux vanishes when $\lambda = \frac{m(b-a)(a^2 \cancel{\;} +4ab+b^2)}{(a+b)^3}$

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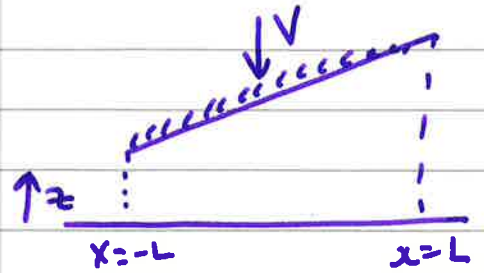
~~In the special case $a=b$, $\lambda=0$, $D=B=0 \Rightarrow w(0)=0$~~

2

~~i.e. asymmetric velocity profile~~

omit

2.



$h = h_0 + \Delta h x/L$

Scaling for other lubrication flows in lectures

(a) For lubrication regime : $h_0/L \ll 1$ (thin gap)

and $\left|\rho u \frac{\partial u}{\partial x}\right| \ll \left|\mu \frac{\partial^2 u}{\partial z^2}\right|$ (negligible inertia)

$\Rightarrow \quad \rho \frac{u}{L}\frac{h^2}{\mu} \ll 1$

but from mass conservation $\frac{u}{L} \sim \frac{V}{h} \Rightarrow \rho \frac{Vh}{\mu} \ll 1$

4

In lectures

(b) Lubrication equation $0 = -\frac{\partial p}{\partial z}$ and $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$

Subjection to $u(0) = u(h) = 0 \rightarrow u = \frac{\partial p}{\partial x}\frac{1}{2\mu}(z^2 - hz)$

Fluid flux per unit width $q = \int_0^h u\, dz = \frac{-h^3}{12\mu}\frac{\partial p}{\partial x}$

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Similar to example in lectures

(c) Mass conservation (pointwise) $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

$\rightarrow \int_0^h \frac{\partial u}{\partial x} dz + [w]_0^h = 0$

$\rightarrow \frac{\partial}{\partial x}\int_0^h u\, dz - u(h)\frac{\partial h}{\partial x} + w(h) = 0$

$\Rightarrow \frac{\partial q}{\partial x} = V$ since $w(h) = -V$

5 Integrating $q(L) - q(-L) = 2VL.$ —— (1)

Unseen

(d) $\frac{\partial p}{\partial x} = -\frac{12\mu q}{h^3}$. Then integrate $\int_{-L}^{L}\frac{dp}{dx}dx = \int_{-L}^{L}\frac{-12\mu q}{h^3}dx$

but no imposed pressure gradient $\Rightarrow p(L) = p(-L) \Rightarrow \int_{-L}^{L}\frac{q}{h^3}dx = 0$

Integrate by parts $\int_{-L}^{L} \frac{q}{(h_0+\Delta h x/L)^3}\, dx$

$$\left[\frac{Lq}{-2\Delta h}\left(h_0+\frac{\Delta h x}{L}\right)^{-2}\right]_{-L}^{L} + \int_{-L}^{L} \frac{L}{2\Delta h}\left(h_0+\frac{\Delta h x}{L}\right)^{-2}\frac{dq}{dx}\,dx = 0$$

$$\Rightarrow \frac{-L}{2\Delta h}\left(\frac{q(L)}{(h_0+\Delta h)^2} - \frac{q(-L)}{(h_0-\Delta h)^2}\right) + \frac{LV}{2\Delta h}\left[-\frac{L}{\Delta h}\left(h_0+\frac{\Delta h x}{L}\right)^{-1}\right]_{-L}^{L} = 0$$

$$\Rightarrow -\left(\frac{q(L)}{(h_0+\Delta h)^2} - \frac{q(-L)}{(h_0-\Delta h)^2}\right) + -\frac{VL}{\Delta h}\left(\frac{1}{h_0+\Delta h} - \frac{1}{h_0-\Delta h}\right) = 0$$

$$\Rightarrow \frac{q(L)}{(h_0+\Delta h)^2} - \frac{q(-L)}{(h_0-\Delta h)^2} = \frac{2VL\Delta h}{\Delta h(h_0+\Delta h)(h_0-\Delta h)}$$

$$\Rightarrow q(L)(h_0-\Delta h)^2 - q(-L)(h_0+\Delta h)^2 = 2VL(h_0+\Delta h)(h_0-\Delta h) \quad \text{—(2)}$$

Solve —(1) and —(2)

$$\Rightarrow q(L)\left[(h_0-\Delta h)^2-(h_0+\Delta h)^2\right] + 2VL(h_0+\Delta h)^2 = 2VL(h_0^2-\Delta h^2)$$

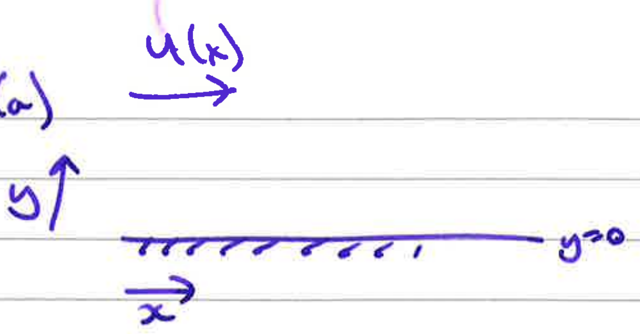
$$\Rightarrow q(L)\cdot -4h_0\Delta h = 2VL(\cancel{h_0^2} - \cancel{\Delta h^2} - \cancel{h_0^2} - 2\Delta h h_0 - \cancel{\Delta h^2})$$

$$= -4VL(h_0\Delta h + \Delta h^2)$$

$$\Rightarrow q(L) = V\frac{L}{h_0}(h_0+\Delta h) = VL\cdot(1+\Delta h/h_0)$$

10 and $q(-L) = -VL(1-\Delta h/h_0)$

3(a)



Results in lectures + derivation

- $U(x)$ does not satisfy no-slip condition on $y=0$
  $\Rightarrow$ boundary layer within which viscous processes reduce U.
- $\underline{u} = (u, v)$

Boundary layer equations (steady)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Justified as follows • $\left|\frac{\partial^2 u}{\partial y^2}\right| \gg \left|\frac{\partial^2 u}{\partial x^2}\right|$ since $1 \gg \left(\delta/L\right)^2$

• $\left|-\frac{1}{\rho}\frac{\partial p}{\partial y}\right| \gg \left|u\frac{\partial v}{\partial x}\right|$ since $\frac{p}{\rho\delta} \gg \frac{UV}{L} = \frac{U^2\delta}{L^2}$

ok since $p \sim \rho U^2 \Rightarrow 1 \gg \delta^2/L^2$

• $\left|-\frac{1}{\rho}\frac{\partial p}{\partial y}\right| \gg \left|\nu\frac{\partial^2 v}{\partial y^2}\right|$ since $\frac{p}{\delta\rho} \gg \frac{\nu V}{\delta^2} = \frac{\nu U}{\delta L}$

ok since $1 \gg \frac{1}{Re}$ $\quad Re = \frac{UL}{\nu}$

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Similar method to examples

(b) Balance viscous and inertial terms to estimate boundary layer thickness

$$\nu U/\delta^2 \sim \frac{U^2}{X} \Rightarrow \delta^2 \sim \frac{\nu X}{U} = \frac{\nu X^2}{\Gamma} \Rightarrow \delta \sim \left(\frac{\nu}{\Gamma}\right)^{1/2} x$$

(c) Introduce streamfunction $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$ $\quad$ Define $\delta = \left(\frac{\nu}{\Gamma}\right)^{1/2} x$

and $\psi = -\frac{\Gamma}{x}\cdot\delta(x)\cdot f\left(\frac{y}{\delta(x)}\right) = -(\nu\Gamma)^{1/2} f\left(\frac{y}{\delta}\right)$. $\quad [\xi = y/\delta]$

$$u = \frac{\partial\psi}{\partial y} = -\frac{(\nu\Gamma)^{1/2}}{\delta}f'. \qquad v = -\frac{\partial\psi}{\partial x} = (\nu\Gamma)^{1/2}\frac{\xi\delta'}{\delta}f' = \frac{(\nu\Gamma)^{1/2}}{x}\xi f'$$

$$\frac{\partial u}{\partial x} = +\frac{(\nu\Gamma)^{1/2}}{\delta^2}\delta' f' + \frac{(\nu\Gamma)^{1/2}}{\delta}\cdot\eta\frac{\delta'}{\delta}f'' = \frac{(\nu\Gamma)^{1/2}}{\delta}\left[+\frac{f'}{x} + \frac{\eta f''}{x}\right]$$

$$\frac{\partial u}{\partial y} = -\frac{(\nu\Gamma)^{1/2}}{\delta^2}f'' \quad , \quad \frac{\partial^2 u}{\partial y^2} = -\frac{(\nu\Gamma)^{1/2}}{\delta^3}f'''$$

Pressure gradient $\quad -\frac{1}{\rho}\frac{dp}{dx} = U.U' = -\frac{\Gamma}{x}\cdot\frac{+\Gamma}{x^2}$

$$-\frac{(\nu\Gamma)^{1/2}}{\delta}f'\cdot\frac{(\nu\Gamma)^{1/2}}{\delta}\left[+\frac{f'}{x} + \frac{\eta f''}{x}\right] + \frac{(\nu\Gamma)^{1/2}}{x}\eta f'\cdot\frac{(\nu\Gamma)^{1/2}}{\delta^2}f'' = -\frac{\Gamma^2}{x^3} - \nu\frac{(\nu\Gamma)^{1/2}}{\delta^3}f'''$$

$$\Rightarrow \quad -\frac{\nu\Gamma}{\frac{\nu}{\Gamma}x^2}\cdot\frac{f'^2}{x} = -\frac{\Gamma^2}{x^3} - \frac{\nu(\nu\Gamma)^{1/2}}{\left(\frac{\nu}{\Gamma}\right)^{3/2}x^3}f'''$$

$$\Rightarrow \quad f'^2 = 1 + f''' \qquad \text{—}(*)$$

Boundary conditions (i) $u(0)=0 \quad \Rightarrow \quad f'(0)=0$
(ii) $v(0)=0$ (automatically satisfied)
(iii) $u \to -\frac{\Gamma}{x} \quad \Rightarrow \quad f' \to 1$ as $\eta \to \infty$

Unseen example

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(d) If $u = -\frac{\Gamma}{x}\left(3\tanh^2(A+B\eta) - 2\right)$ then $f' = 3\tanh^2(A+B\eta) - 2$

So $\quad f'' = 6B\tanh(..)\operatorname{sech}^2(...) = 6B\left[\tanh(...) - \tanh^3(...)\right]$

$\quad f''' = 6B^2\operatorname{sech}^2(...) - 18B^2\tanh^2(..)\operatorname{sech}^2(...)$

$\quad\quad = 6B^2(1-\tanh^2(..)) - 18B^2\tanh^2(..)(1-\tanh^2(...))$

Substitute into —(*) : $18B^2\tanh^4(...) - 24B^2\tanh^2(...) + 6B^2$
$\quad - (3\tanh^2(...) - 2)^2 + 1 = 0$

$\Rightarrow (18B^2 - 9)\tanh^4(...) + (12 - 24B^2)\tanh^2(...) + 6B^2 - 4 + 1 = 0$

$\Rightarrow B^2 = 1/2 \quad \Rightarrow \quad B = 1/\sqrt{2}$

Checking boundary conditions $f' \to 1$ as $\eta \to \infty$ ✓

$f'(0) = 0 \quad \Rightarrow \quad 3\tanh^2 A = 2$
$\quad\quad \Rightarrow \quad A = \tanh^{-1}\left(\frac{2}{3}\right)^{1/2}$

Unseen example, but straightforward method
close to Proudman in places

6

4 (a) Vorticity $\underline{\omega} = \nabla \wedge \underline{u}$

$$[\underline{u} \wedge \underline{\omega}]_i = \varepsilon_{ijk} u_j \varepsilon_{klm} \nabla_l u_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \nabla_l u_m$$
$$= u_j \nabla_i u_j - u_j \nabla_j u_i$$
$$= \nabla_i (|\underline{u}|^2/2) - \underline{u}.\nabla u_i$$

Euler equation $\quad \dfrac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} = -\frac{1}{\rho}\nabla p$

$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} = \underline{u} \wedge \underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right)$$
$$\Rightarrow \quad \frac{\partial \underline{\omega}}{\partial t} = \nabla \wedge (\underline{u} \wedge \underline{\omega})$$

$$[\nabla \wedge (\underline{u} \wedge \underline{\omega})]_i = \varepsilon_{ijk} \nabla_j \varepsilon_{klm} u_l \omega_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \nabla_j u_l \omega_m$$
$$= \nabla_j u_i \omega_j - \nabla_j u_j \omega_i$$
$$= (\underline{\omega}.\nabla \underline{u})_i - [(\underline{u}.\nabla)\underline{\omega}]_i$$

5 Hence Vorticity equation $\quad \dfrac{\partial \underline{\omega}}{\partial t} + \underline{u}.\nabla \underline{\omega} = \underline{\omega}.\nabla \underline{u}$

For 2-D flow $\quad \underline{\omega}.\nabla \underline{u} \equiv 0 \quad \Rightarrow \quad \dfrac{\partial \underline{\omega}}{\partial t} + \underline{u}.\nabla \underline{\omega} = 0$

2

Results given in lectures

(b) Vorticity $\quad \underline{\omega} = \nabla \wedge V(r)\hat{\underline{\theta}} = \left(\dfrac{\partial V}{\partial r} + \dfrac{V}{r}\right)\hat{\underline{z}} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(rV)\hat{\underline{z}} \equiv \Omega \hat{\underline{z}}$

2

Easy

(c) $\underline{u}_1 = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}\hat{\underline{r}} - \dfrac{\partial \psi}{\partial r}\hat{\underline{\theta}} \quad$ Vorticity $\nabla \wedge \underline{u}_1 = -\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \psi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \psi}{\partial \theta^2}\right)\hat{\underline{z}}$

Vorticity $= (\Omega - \nabla^2 \psi)\hat{\underline{z}} \qquad |\Omega| \gg |\nabla^2 \psi|$ (small perturbation)

$$\frac{\partial}{\partial t}(\Omega - \nabla^2\psi) + \left(V(r)\hat{\underline{\theta}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underline{r}} - \frac{\partial \psi}{\partial r}\hat{\underline{\theta}}\right).\nabla(\Omega - \nabla^2\psi) = 0$$

Linearise to $\quad \dfrac{\partial}{\partial t}(-\nabla^2\psi) - V(r)\hat{\underline{\theta}}.\nabla(\nabla^2\psi) + \left(\dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}\hat{\underline{r}} - \dfrac{\partial \psi}{\partial r}\hat{\underline{\theta}}\right).\nabla\Omega = 0$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} + \frac{V}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) - \frac{d\Omega}{dr}\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

8

Unseen example but cartesian coords in lectures

(d) write $\psi = \phi(r)\exp(st + in\theta)$

$$\left(s + in\frac{V}{r}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) - \frac{n^2}{r^2}\phi\right) = \frac{1}{r}\frac{d\Omega}{dr}in\phi$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) - \frac{n^2}{r^2}\phi = \frac{in}{s+inV/r}\frac{1}{r}\frac{d\Omega}{dr}\phi$$

Multiply by ϕ^*

$$\int_{R_1}^{R_2}\phi^*\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)dr - n^2\int_{R_1}^{R_2}\frac{|\phi|^2}{r}dr = \int_{R_1}^{R_2}\frac{in}{s+inV/r}\frac{d\Omega}{dr}|\phi|^2 dr \quad \text{—}(*)$$

Use integration by parts

$$\int_{R_1}^{R_2}\phi^*\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)dr = \cancel{\left[\phi^* r\frac{\partial\phi}{\partial r}\right]_{R_1}^{R_2}} - \int_{R_1}^{R_2} r\left|\frac{\partial\phi}{\partial r}\right|^2 dr$$

since $\phi(R_1) = \phi(R_2) = 0$

Write $s = s_r + is_I$ $\qquad \dfrac{in\,(s_r - is_I - inV/r)}{s_r^2 + (s_I + nV/r)^2} = \dfrac{in}{s+inV/r}$

Take $\mathrm{Im}\{\text{—}(*)\}$ $\Rightarrow$ $\quad s_r\displaystyle\int_{R_1}^{R_2}\frac{1}{|s+inV/r|^2}\cdot\frac{d\Omega}{dr}|\phi|^2 dr = 0$

For instability, require $s_r > 0$

$\Rightarrow \quad \dfrac{d\Omega}{dr} = 0$ at some point in $R_1 < r < R_2$

Unseen, but quite similar to Cartesian result

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