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Advanced Fluid Dynamics

Preliminaries

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Suffix notation and summation convention

- Suppose we have two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, with the components expressed relative to an orthonormal basis. Then the dot product is given by

$$\mathbf{u}.\,\mathbf{v} = \sum_{i=1}^{3} u_i v_i \equiv u_i v_i$$

In the latter expression we adopt the **summation convention** that repeated suffices imply summation.

Vector products

- Two useful quantities are:

  1. The Kronecker delta $\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$

  2. The antisymmetric symbol ϵ_{ijk}
     $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$
     $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$
     and $\epsilon_{ijk} = 0$ otherwise.

- Dot product $\mathbf{u}.\,\mathbf{v} = \delta_{ij} u_i v_j = u_i v_i$.

- Cross product $[\mathbf{u} \wedge \mathbf{v}]_i = \epsilon_{ijk} u_j v_k$.

Double product

- Important double product: $\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{kp}\delta_{jq}$

- Triple vector product:

$$
\begin{aligned}
[\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})]_i &= \epsilon_{ijk} u_j \epsilon_{kpq} v_p w_q \\
&= \epsilon_{kij} \epsilon_{kpq} u_j v_p w_q \\
&= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) u_j v_p w_q \\
&= u_j v_i w_j - u_j v_j w_i \\
&= [(\mathbf{u}.\,\mathbf{w})\mathbf{v} - (\mathbf{u}.\,\mathbf{v})\mathbf{w}]_i
\end{aligned}
$$

Basis transformations

- A vector is independent of the basis vectors with which it may be represented. For orthonormal bases $\mathbf{e_i}$ and $\mathbf{e_i}'$

$$\mathbf{v} = \sum_{i=1}^{3} a_i \mathbf{e_i} = \sum_{i=1}^{3} a_i' \mathbf{e_i}'$$

- $a_i' = \sum_{j=1}^{3} a_j \mathbf{e_i}' . \mathbf{e_j} = l_{ij} a_j$, where l_{ij} are the direction cosines (and summation assumed).

- A second order tensor, σ_{ij}, also has the property of invariance to basis vectors. Thus is transforms according to

$$\sigma_{ij}' = l_{ip} l_{jq} \sigma_{pq}.$$

Tensors

- Tensors are important for mathematical expressions of physical laws - such laws can not depend on the coordinate systems and thus must be formulated in terms of tensors.

- Isotropic tensors are the same in all orthonormal coordinates systems.

- The second order isotropic tensor is $\alpha\delta_{ij}$ (α constant).

- Isotropic tensors have no preferred orientations.

Derivatives of vector fields

For suitably diffentiable scalar fields, $\phi(\mathbf{x})$ and vector fields, $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$

- $\mathrm{div}\mathbf{u} \equiv \nabla.\,\mathbf{u} = \frac{\partial u_i}{\partial x_i}$
- $\mathrm{curl}\mathbf{u} \equiv \nabla \wedge \mathbf{u} = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j}$
- $\mathrm{grad}\mathbf{u} \equiv [\nabla \mathbf{u}]_{ij} = \frac{\partial u_j}{\partial x_i}$
- $\nabla \wedge \nabla\phi = 0$
- $\nabla.\,\nabla \wedge \mathbf{u} = 0$

Integrals

- Divergence Theorem: if V is a simply connected domain with surface S an outward pointing unit normal $\mathbf{n}$, then

$$\int_V \nabla . \mathbf{u}\, \mathrm{d}V = \int_S \mathbf{u} . \mathbf{n}\, \mathrm{d}S$$

- Stokes's Theorem: if C is a directed closed curve, spanned by a surface S then

$$\oint_C \mathbf{u} . \mathrm{d}\mathbf{l} = \int_S \nabla \wedge \mathbf{u} . \mathbf{n} \mathrm{d}S.$$

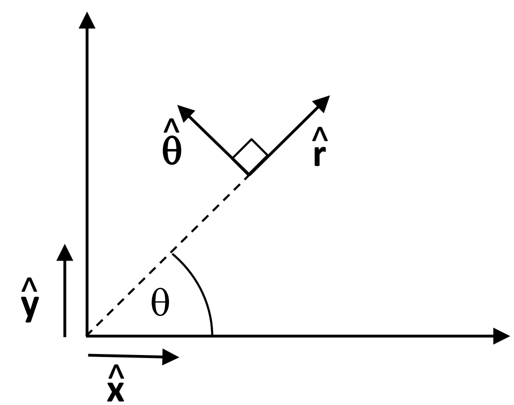
Cylindrical polar coordinates

- $x = r\cos\theta$, $y = r\sin\theta$ and Jacobian: $\left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$.

- Basis vectors: $\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\end{aligned}$

- $\dfrac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}}$ and $\dfrac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{r}}$

- Gradient: $$\begin{aligned}\nabla &\equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \\ &= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\end{aligned}$$



Laplacian

$$
\begin{aligned}
\nabla^2\phi &= \nabla . \nabla\phi \\
&= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) . \left(\hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}\right) \\
&= \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}.
\end{aligned}
$$