1 Governing equations of fluid motion

1.1 Terminology

There are three common states of matter, gas, liquid and solid, and the term **fluid** encompasses both liquids and gases. The distinction between solids and fluids is the ease with which they deform. A *simple fluid* is immediately deformed by a shear force, whereas a *simple solid* may resist this deformation. Important examples of simple fluids are water and air.
A *continuum model* is used to describe the motion of fluids. This does not represent dynamics at the molecular level, but rather at some much larger lengthscale.
Usually dependent variables are modelled using a **Eulerian** description and are treated as function of space $\mathbf{x}$ and time t. The rate of change of a quantity following a fluid element is the **material derivative**

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u}.\nabla \tag{1}$$

This is often denoted by D/Dt.

1.2 Conservation of mass

The mass (M) contained within an arbitrary volume V, fixed in space, with boundary S and outward pointing unit normal, $\mathbf{n}$, is given by

$$M = \int_V \rho \, \mathrm{d}V, \tag{2}$$

where $\rho(\mathbf{x}, t)$ is the fluid density. The mass can only change by transport of fluid over the bounding surface because there are no sources/sinks enclosed. Thus

$$\frac{\mathrm{d}M}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = -\int_S \rho\mathbf{u}.\mathbf{n} \, \mathrm{d}S. \tag{3}$$

Simplifying using the divergence theorem yields

$$\int_V \frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{u}) \; \mathrm{d}V = 0. \tag{4}$$

But the volume V is arbitrary and so the integrand must vanish and this gives the pointwise expression of mass conservation as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{u}) = 0. \tag{5}$$

Equivalently this may be written

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} = 0. \tag{6}$$

1.3 Rate of strain tensor

Close to $\mathbf{x}_0$, the velocity field may be expanded using a Taylor series as

$$u_i(\mathbf{x}) - u_i(\mathbf{x}_0) = (x_j - x_{0j}) \left.\frac{\partial u_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0} + \dots \tag{7}$$

and the velocity gradient tensor $\partial u_i/\partial x_j$ may be written as the sum of a symmetric and anti-symmetric tensor.

$$\left.\frac{\partial u_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \tag{8}$$

$$= e_{ij}+\Omega_{ij}, \tag{9}$$

where e_{ij} and Ω_{ij} are termed the *rate of strain* and *vorticity* tensors, respectively. The rate of strain tensor is symmetric $e_{ij}=e_{ji}$ and the vorticity tensor is antisymmetric $\Omega_{ij}=-\Omega_{ji}$. The vorticity tensor can be related to the vector Ω_k by $\Omega_{ij}=\epsilon_{ikj}\Omega_k$ and the vorticity of the flow $\boldsymbol{\omega}=\nabla\wedge\mathbf{u}=2\boldsymbol{\Omega}$.
Example: for simple shear flows, $\mathbf{u}=(\gamma y,0,0)$,

$$\frac{\partial u_i}{\partial x_j}=\begin{pmatrix}0&\gamma&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&\gamma/2&0\\\gamma/2&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&\gamma/2&0\\-\gamma/2&0&0\\0&0&0\end{pmatrix}. \tag{10}$$

Thus $\mathbf{u}$ may be interpreted as the sum of a straining flow $\mathbf{u}_1=(\gamma y/2,\gamma x/2,0)$ and a solid body rotation $\mathbf{u}_2=(\gamma y/2,-\gamma x/2,0)=(0,0,-\gamma/2)\wedge(x,y,z)$. Note that the straining flow leads to the separation of adjacent fluid elements, whereas the solid body rotation does not.

1.4 Stress tensor

Fluids experience both 'body' and 'surface forces'. Body forces are long range, penetrate the interior of the fluids and act on all fluid elements. Examples include gravity and electromagnetic forces. Over a small volume, δV, the force is given by $\mathbf{F}_V(\mathbf{x},t)\delta V$. Surface forces are relatively short range and have a direct molecular origin. They are proportional to the surface area of between two fluid elements, δA, and are a function of space, time and possibly the orientation of the surface, here specified by a unit normal $\mathbf{n}$. It is thus given by $\mathbf{F}_S(\mathbf{x},t,\mathbf{n})\delta A$.

By Newton's third law, we must have $\mathbf{F}_S(\mathbf{x},t,\mathbf{n})=-\mathbf{F}_S(\mathbf{x},t,-\mathbf{n})$. This implies that $\mathbf{F}_S$ is an odd function of $\mathbf{n}$.

We consider a small tetrahedron with vertices $OABC$, where the vectors OA, OB and OC are denoted by the mutually orthogonal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$, respectively. Since the tetrahedron is sufficiently small, body forces are negligible and in equilibrium, the surface forces must balance.

$$\text{Force on OAB} = \mathbf{F}_S(\mathbf{x},t,-\mathbf{n}_3)\text{Area OAB} \tag{11}$$

$$= \mathbf{F}_S(\mathbf{x},t,-\mathbf{n}_3)\frac{1}{2}\frac{\mathbf{n}_3}{|\mathbf{n}_3|}.\,\mathbf{n}_1\wedge\mathbf{n}_2. \tag{12}$$

But if the area of ABC is denoted by δA, $-\mathbf{n}\delta A=\frac{1}{2}(\mathbf{n}_3-\mathbf{n}_1)\wedge(\mathbf{n}_2-\mathbf{n}_1)$ and thus $\frac{1}{2}\mathbf{n}_3.\,\mathbf{n}_1\wedge\mathbf{n}_2=\mathbf{n}_3.\,\mathbf{n}\delta A$ and so

$$\text{Force on OAB} = \mathbf{F}_S(\mathbf{x},t,-\mathbf{n}_3)\frac{\mathbf{n}_3}{|\mathbf{n}_3|}.\,\mathbf{n}\delta A. \tag{13}$$

The forces on OBC and OAC can be expressed analogously and thus the force balance is given by

$$\mathbf{F}_S(\mathbf{x},t,\mathbf{n})\delta A+\mathbf{F}_S(\mathbf{x},t,-\mathbf{n}_3)\frac{\mathbf{n}_3}{|\mathbf{n}_3|}.\,\mathbf{n}\delta A+\mathbf{F}_S(\mathbf{x},t,-\mathbf{n}_2)\frac{\mathbf{n}_2}{|\mathbf{n}_2|}.\,\mathbf{n}\delta A+\mathbf{F}_S(\mathbf{x},t,-\mathbf{n}_1)\frac{\mathbf{n}_1}{|\mathbf{n}_1|}.\,\mathbf{n}\delta A=0. \tag{14}$$

Since this can not depend upon th choice of basis vectors, we deduce

$$F_{Si}(\mathbf{n})=\sigma_{ij}n_j, \tag{15}$$

where σ_{ij} is the stress tensor. σ_{ij} represents the i^{th} component of force exerted across a plane surface normal to the direction j at a position $\mathbf{x}$ at a time t. It is independent of $\mathbf{n}$ - it depends only on properties of the fluid motion.
The stress tensor is symmetric $\sigma_{ij} = \sigma_{ji}$. This is deduced by considering the torque exerted on the small tetrahedron

$$\begin{aligned} \text{Torque } G_i &= \epsilon_{ijk} \int_S x_j F_{Sk} \, \mathrm{d}S = \epsilon_{ijk} \int_S x_j \sigma_{kl} n_l \, \mathrm{d}S && (16) \\ &= \epsilon_{ijk} \int_V \frac{\partial}{\partial x_l} \left(x_j \sigma_{kl} \right) \, \mathrm{d}V = \epsilon_{ijk} \int_V \sigma_{kj} + x_j \frac{\partial \sigma_{kl}}{\partial x_l} \, \mathrm{d}V && (17) \end{aligned}$$

The first integral of this final expression is much larger than the second as the size of the tetrahedron becomes small. Thus if the tetrahedron is torque-free, we deduce

$$\epsilon_{ijk} \sigma_{kj} = 0 \tag{18}$$

and from this we deduce directly that the stress tensor is symmetric.

We write the stress tensor as follows

$$\sigma_{ij} = -p \delta_{ij} + \sigma'_{ij}, \tag{19}$$

where the pressure is given by $p = -\frac{1}{3}\sigma_{kk}$ and the deviatoric stress tensor is denoted by σ'_{ij}. The latter is trace-free by construction.

1.5 The relationship between stress and rate of strain

In general a constitutive relationship provides the connection between the deviatoric stress tensor and the rate of strain tensor. The latter provides a measurement of the rate of deformation of the fluid elements. Importantly this is not a function of the velocity field - but rather its gradient - because the velocity would change with the frame of reference. For simple fluids ('Newtonian fluids') such as water and air, it is found experimentally that

- σ'_{ij} is a linear function of $\frac{\partial u_i}{\partial x_j}$
- σ'_{ij} does not depend on displacement of fluid elements (ie no elastic effects)
- There are no 'memory' effects - the relationship is instantaneous
- The relationship is isotropic.

We therefore pose a linear relationship between the stress and velocity gradient tensors, which can be written

$$\sigma'_{ij} = A_{ijkl} \frac{\partial u_k}{\partial x_l}, \tag{20}$$

where $A_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$ (α, β, and γ constants) is a fourth order isotropic tensor. The requirements that the deviatoric stress tensor is symmetric and trace-free lead to

$$\sigma_{ij} = -p \delta_{ij} + 2\mu e_{ij} - \frac{2}{3} \mu e_{kk} \delta_{ij}, \tag{21}$$

where e_{ij} is the symmetric rate of strain tensor (see §1.3) and μ is the dynamic viscosity, which is a material property. The pressure p in the static state may be determined from thermodynamic laws. In the flowing state it is potentially different - but this difference is negligible and will henceforth be neglected.

1.6 Conservation of momentum

The momentum of the fluid within an arbitrary volume V with bounding surface S and outward point normal $\mathbf{n}$ is $\int_V \rho\mathbf{u}\,\mathrm{d}V$. Momentum changes due to transport across the surface S and the action of body forces $\mathbf{F}$ and surface stresses σ_{ij}. Thus momentum balance is expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = -\int_S \rho\mathbf{u}\mathbf{u}.\,\mathbf{n}\,\mathrm{d}S + \int_V \mathbf{F}\,\mathrm{d}V + \int_S \sigma.\,\mathbf{n}\,\mathrm{d}S. \tag{22}$$

The volume V is fixed and on applying the divergence theorem

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\;\mathrm{d}V = \int_V -\frac{\partial}{\partial x_j}(\rho u_i u_j) + F_i + \frac{\partial \sigma_{ij}}{\partial x_j}\,\mathrm{d}V. \tag{23}$$

Since the volume V is arbitrary, the integrand must vanish and on simplifying using the expression for mass conservation, we obtain the **Cauchy momentum equation**

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i. \tag{24}$$

Finally for a Newtonian fluid we obtain

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\,\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \frac{1}{3}\mu\nabla\left(\nabla.\,\mathbf{u}\right) + \mathbf{F}. \tag{25}$$

1.7 Boundary conditions

1. The normal velocity $(\mathbf{u}.\,\mathbf{n})$ at a boundary/interface is continuous.

2. At a solid boundary, the tangential velocity is continuous. Denoting the boundary velocity by $\mathbf{V}$, this implies that

$$\mathbf{u} - (\mathbf{u}.\,\mathbf{n})\mathbf{n} = \mathbf{V} - (\mathbf{V}.\,\mathbf{n})\mathbf{n}. \tag{26}$$

   But since the normal velocity is continuous, this implies that $\mathbf{u} = \mathbf{V}$. This is known as the **no-slip boundary condition**.

3. At a deformable interface in the absence of surface tension, the stress field is continuous $\sigma_{ij}n_j$. Thus there are continuity conditions on both the normal stress and the tangential stress. (The latter is often termed the shear stress.)

1.8 Incompressibility

A fluid is incompressible if the density of a fluid element does not change following the fluid element. This implies that

$$\frac{D\rho}{Dt} = 0. \tag{27}$$

Then directly from mass conservation (6), we deduce that $\nabla.\,\mathbf{u} = 0$.

This approximation is good for: (i) flow speeds much less than the speed of sound; (ii) low frequency motion; and (ii) relatively small changes in height so that there is no gravitational compression.

Thus the **incompressible Navier-Stokes equations** are

$$\nabla.\,\mathbf{u} = 0, \tag{28}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{F}. \tag{29}$$

The pressure in this system is not determined from an equation of state relating local conditions, but rather from the global need to satisfy $\nabla.\, \mathbf{u} = 0$.

The **vorticity** of the flow is defined by $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$. The vorticity equation is derived by taking the curl of the momentum equation (29). If there are no body forces ($\mathbf{F} = 0$) and the density is constant then upon simplifying this gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}.\, \nabla \boldsymbol{\omega} = \boldsymbol{\omega}.\, \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{30}$$

Here the kinematic viscosity $\nu = \mu/\rho$. The terms may be readily interpreted: $\mathbf{u}.\, \nabla \boldsymbol{\omega}$ represents advection of vorticity by the flow; $\boldsymbol{\omega}.\, \nabla \mathbf{u}$ represents stretching of the velocity field; and $\nu \nabla^2 \boldsymbol{\omega}$ viscous diffusion of vorticity.

1.9 Energy equation for incompressible flows

The energy density is defined to be $E = \frac{1}{2}\rho \mathbf{u}.\, \mathbf{u}$. Its evolution equation can be derived by taking the dot product between the velocity field and the momentum equation. Then we find

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} \left(E u_j - u_i \sigma_{ij} \right) = -2\mu e_{ij} e_{ij} + F_i u_i. \tag{31}$$

We integrate this over a volume V with bounding surface S and unit outward pointing normal $\mathbf{n}$ to find

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V E \,\mathrm{d}V + \int_S E(\mathbf{u}.\, \mathbf{n}) \,\mathrm{d}S = \int_S u_i \sigma_{ij} n_j \,\mathrm{d}S - 2\mu \int_V e_{ij} e_{ij} \,\mathrm{d}V + \int_V \mathbf{u}.\, \mathbf{F} \,\mathrm{d}V. \tag{32}$$

The terms in this equation respectively correspond to the rate of change of kinetic energy; the transport of kinetic energy; the work done by surface stresses ($\tau_i = \sigma_{ij} n_j$); viscous dissipation; and the work done by body forces.

Importantly $-2\mu \int_V e_{ij} e_{ij} \,\mathrm{d}V \leq 0$ and so viscous processes always cause the kinetic energy to decay. Thus steady flows can only be maintained if there is energy input to match this dissipation.