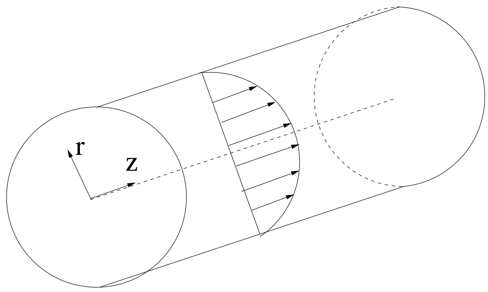
2 Simple flow fields

2.1 Poiseuille flow: *Fully developed, steady pipe flow*



We construct the steady axial flow of fluid of dynamic viscosity μ along a pipe of radius a. It is convenient to adopt cylindrical polar coordinates (r, θ, z), with the z-axis aligned with the axis of the pipe. The flow field is independent of angle by the symmetry of the problem and thus we seek a flow field of the form $\mathbf{u} = w(r)\hat{\mathbf{z}}$.

Mass conservation is automatically satisfied as $\nabla . \mathbf{u} = \frac{\partial w}{\partial z} = 0$.

The axial component of the momentum equation demands that

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \tag{1}$$

while the radial component gives $\partial p/\partial r = 0$. Thus the pressure is only a function of axial distance and the pressure gradient is constant. We write $\partial p/\partial z = -\Delta P/L$, where $\Delta P > 0$ is the pressure drop between parts of the pipe separated by an axial distance L. Integrating (1), applying the no-slip boundary condition on the pipe boundary ($w(a) = 0$) and requiring that $w(r)$ remains bounded at $r = 0$, we find

$$w = -\frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\left(a^2 - r^2\right). \tag{2}$$

The volume flux of fluid transport along the pipe, Q, is given by

$$Q = \int_0^a w 2\pi r \,\mathrm{d}r = \frac{\pi \Delta P a^4}{8\mu L}. \tag{3}$$

The tangential shear stress exerted on the wall σ_{rz} is then given by

$$\sigma_{rz} = 2\mu e_{rz}(r = a) = \mu \left.\frac{\partial w}{\partial r}\right|_{r=a} = -\frac{\Delta P a}{2L}. \tag{4}$$

The rate of viscous dissipation is given by

$$2\mu \int_V e_{ij}e_{ij}\,\mathrm{d}V = \mu \int_0^L \int_0^{2\pi} \int_0^a \left(\frac{\partial w}{\partial r}\right)^2 r \,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = \frac{\pi \Delta P^2 a^4}{8\mu L} = Q\Delta P. \tag{5}$$

2.1.1 Dimensional analysis: *Reynolds experiment*

What pressure gradient is required to drive a volume flux of fluid, Q, of density ρ and viscosity μ along a pipe of radius a?

This may be addressed using dimensional analysis to write the pressure gradient $[\partial p/\partial z] = ML^{-2}T^{-2}$ as a function of the velocity, $U = Q/a^2$, $[U] = LT^{-1}$, radius $[a] = L$, density $[\rho] = ML^{-3}$ and viscosity $[\mu] = ML^{-1}T^{-1}$.

Dimensional analysis asserts that a dimensionally consistent relationship will feature a function of one dimensionless ratio because there are 3 independent dimensions, but 4 independent parameters. Thus we may write

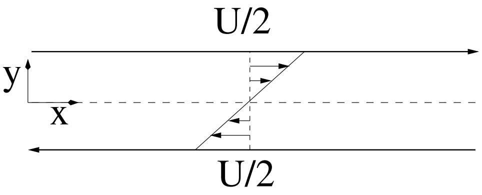
$$-\frac{\partial p}{\partial z} = \frac{\rho U^2}{a} F\left(\frac{\rho U a}{\mu}\right), \tag{6}$$

where F is an undetermined function and $Re = \rho U a/\mu$ is a dimensionless ratio, known as the *Reynolds number*. Poiseuille flow has established that

$$-\frac{\partial p}{\partial z} = \frac{8U\mu}{\pi a^2} = \frac{\rho U^2}{a}\frac{8\mu}{\pi a U \rho}. \tag{7}$$

Thus $F(Re) = 8/(\pi Re)$. Reynolds found that this law was good for $Re < 1000$. For larger Reynolds numbers, the flow becomes unsteady and turbulent.

2.2 Couette flow: *flow between parallel translating plates*



We consider two parallel planes, $y = a/2$ and $y = -a/2$, which are translating at velocities $\mathbf{u} = \pm(U/2)\hat{\mathbf{x}}$ and determine the flow field. The fluid motion is driven only by the translating plates; there are no pressure gradients.

We seek a solution of the form $\mathbf{u} = u(y)\hat{\mathbf{x}}$. This satisfies mass conservation automatically ($\nabla.\mathbf{u} = 0$), while the x component of the momentum gives

$$0 = \mu\frac{\partial^2 u}{\partial y^2}. \tag{8}$$

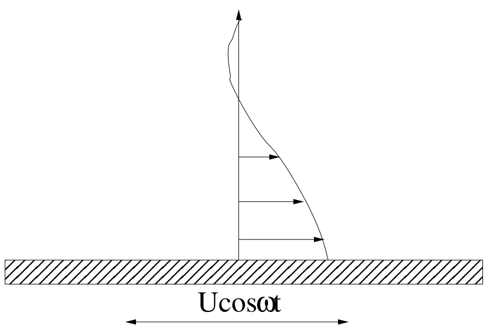
Integrating and applying no-slip boundary conditions at $y = \pm a/2$, we find

$$u(y) = \frac{Uy}{a}. \tag{9}$$

The shear stress $\sigma_{xy} = \mu \partial u/\partial y = \mu U/a$. Thus the stress exerted on the upper plate where the outward pointing normal $\mathbf{n} = \hat{\mathbf{y}}$ is $\boldsymbol{\tau} = \frac{\mu U}{a}\hat{\mathbf{x}}$. The stress exerted on the lower plate where the outward pointing normal $\mathbf{n} = -\hat{\mathbf{y}}$ is $\boldsymbol{\tau} = -\frac{\mu U}{a}\hat{\mathbf{x}}$.

The rate of viscous dissipation per unit area is given by $2\mu\int_V e_{ij}e_{ij}\,\mathrm{d}V = 2\mu\int_{-a/2}^{a/2}\frac{U^2}{2a^2}\,\mathrm{d}y = \frac{\mu U^2}{a}$. This is balanced by the rate at which work is done at the boundaries ($\boldsymbol{\tau}.\mathbf{u}$).

2.3 Oscillating flat plate



A flat plate at $y = 0$ persistently oscillates and drives fluid motion in the overlying semi-infinite domain. The plate motion is given by $U\cos\omega t\hat{\mathbf{x}}$ and there is no pressure gradient as the fluid motion is solely driven by the oscillations of the plate. We seek a solution of the form $\mathbf{u} = u(y,t)\hat{\mathbf{x}}$. This automatically satisfies $\nabla.\mathbf{u} = 0$. The x-component of the momentum equation becomes

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{10}$$

which subject to boundary conditions $u(0,t) = U\cos\omega t$ and $u(y,t) \to 0$ as $y \to \infty$. The governing equation (10) is a diffusion equation with the kinematic viscosity playing the role of a diffusivity.

We look for a solution by writing $u = \hat{u}(y)\mathrm{e}^{\mathrm{i}\omega t}$ [take real part of solution]. This leads to

$$\frac{\mathrm{d}^2\hat{u}}{\mathrm{d}y^2} - \frac{\mathrm{i}\omega}{\nu}\hat{u} = 0, \tag{11}$$

and noting that $\sqrt{\mathrm{i}} = \pm(1+\mathrm{i})/\sqrt{2}$ and applying the boundary conditions, we find that the velocity field is given by

$$u(y,t) = U\mathrm{e}^{-y\sqrt{\frac{\omega}{2\nu}}}\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right). \tag{12}$$

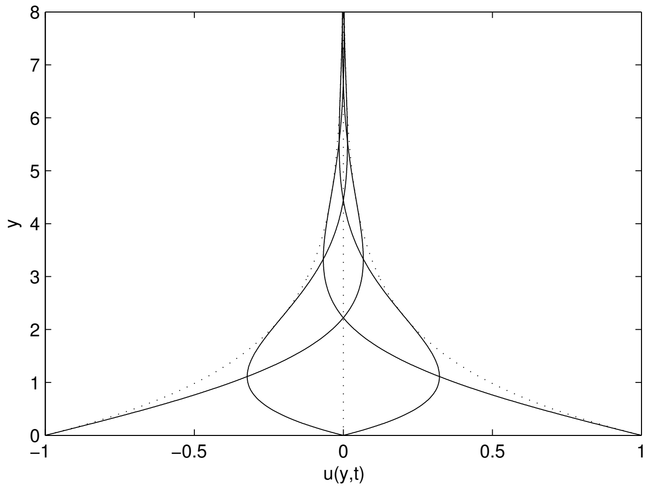


Figure 1: The velocity field, $u(y,t)$, above an oscillating flat plate with velocity amplitude $U = 1$ and viscosity $\nu = 1$ at times $t = \pi/2, \pi, 3\pi/2$ and 2π. The dotted line shows the envelope of the maximum velocities at each elevation.

The velocity field decays exponentially with distance from the plate $|u(y,t)| = U\mathrm{e}^{-y\sqrt{\frac{\omega}{2\nu}}}$. The lengthscale of the decay is $\sqrt{2\nu/\omega}$, which is approximately 1mm in water oscillating at 1Hz. The shear stress at the plate $\sigma_{xy} = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = -\mu U\sqrt{\frac{\omega}{\nu}}\cos\left(\omega t + \frac{\pi}{4}\right)$. It is thus $\pi/4$ out of phase with the velocity field.

The time average viscous dissipation is given by

$$\frac{\omega}{2\pi}\int_0^{\pi/\omega}\int_0^{\infty}\mu\left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}y = \frac{1}{2\sqrt{2}}\mu U^2\sqrt{\frac{\omega}{\nu}}. \tag{13}$$

It may be shown that this is equal to the time average rate of working of the stress at the boundary, namely

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} u(0,t)\sigma_{xy}(0,t)\,\mathrm{d}t. \tag{14}$$

2.4 Impulsively started flat plate

A flat plat is initially at rest and lies along the plane $y = 0$, with a semi-infinite fluid layer above it. At $t = 0$, it is impulsively started and begins to move at constant velocity $\mathbf{u} = U\hat{\mathbf{x}}$.

We seek the velocity field for the fluid motion of the form $\mathbf{u} = u(y,t)\hat{\mathbf{x}}$. This field automatically satisfies $\nabla.\mathbf{u} = 0$ and since there is no imposed pressure gradient, the x-momentum equation is given by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{15}$$

subject to $u(0,t) = U$ for $t > 0$ and $u = 0$ at $t = 0$. This governing equation is a diffusion equation. It could be solved using Laplace transforms, but here we construct the solution directly by seeking a similarity solution. Thus we pose $u = UF(y/\sqrt{\nu t})$ for an as yet undetermined function F. On substitution into (15) and by writing $\eta = (y/\sqrt{\nu t})$, we find that

$$-\frac{\eta U}{2t} F' = \frac{U}{t} F''. \tag{16}$$

This may be integrated directly to give $F = B - A \int_\eta^\infty \mathrm{e}^{-s^2/4}\, \mathrm{d}s$, where A and B are constants. Enforcing $u = 0$ at $t = 0$, implies that $F \to 0$ as $\eta \to \infty$. Thus $B = 0$. Then from $u(0,t) = U$, we deduce that

$$1 = -A \int_0^\infty \mathrm{e}^{-s^2/4}\, \mathrm{d}s, \quad \text{which implies} \quad A = -\frac{1}{\sqrt{\pi}}. \tag{17}$$

The flow field is then given by

$$u = \frac{U}{\sqrt{\pi}} \int_{y/\sqrt{\nu t}}^\infty \mathrm{e}^{-s^2/4}\, \mathrm{d}s = U \mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{18}$$

where $\mathrm{erfc}(x)$ is the complimentary error function.

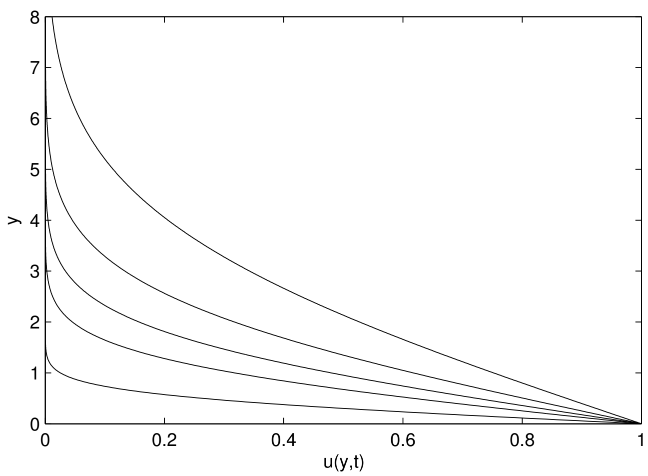


Figure 2: The velocity field, $u(y,t)$, for an impulsively started flat plate with velocity $U = 1$ and viscosity $\nu = 1$ at times $t = 0.1, 0.5, 1, 2$ and 5.

This flow field is usefully considered in terms of its vorticity

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} \mathrm{e}^{-y^2/(4\nu t)}. \tag{19}$$

Initially there is no flow and so the vorticity vanishes throughout the domain. For $t > 0$, vorticity is generated at the boundary (by virtue of the no-slip condition) and diffuses into the domain. The lengthscale over which diffusion occurs is when $y \sim \sqrt{\nu t}$.

2.5 Burger's vortex

We consider flow that is purely in an angular direction. Adopting cylindrical polar coordinates, we write the velocity field as $\mathbf{u} = v(r,t)\hat{\boldsymbol{\theta}}$ and so mass conservation is automatically satisfied $(\nabla.\,\mathbf{u} = 0)$.

The radial component of the velocity field gives

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{20}$$

This implies that the gradient of the pressure field supplies the force that ensures circular motion.

The angular component of the velocity field gives

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right). \tag{21}$$

This is the balance of angular momentum as the viscous forces exert torques on the the fluid. Initially there is an inviscid line vortex at $r = 0$, given by $v = C/(2\pi r)$, where C is the circulation about it. We now integrate (21) to find the solution in $t > 0$.

We look for a similarity solution of the form $v = \dfrac{C}{\pi r}F\left(\dfrac{r}{\sqrt{\nu t}}\right)$. On substituting into (21), writing $\eta = r/\sqrt{\nu t}$ and simplifying, we find

$$2\eta F'' - (2-\eta^2)F' = 0. \tag{22}$$

This may be integrated subject to the initial condition that $F = 1$ at $t = 0$ and that v is regular at $r = 0$ for $t > 0$. Thus we find that

$$v(r,t) = \frac{C}{2\pi r}\left(1 - \mathrm{e}^{-r^2/(4\nu t)}\right). \tag{23}$$

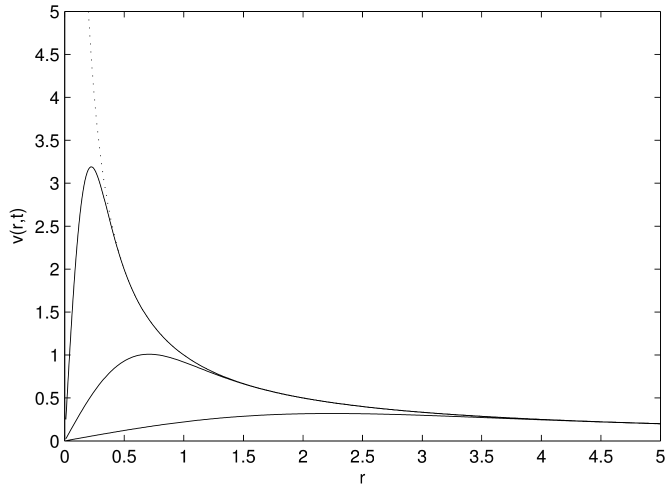


Figure 3: The velocity field, $v(r,t)$, for the Burger's vortex with circulation $C = 2\pi$, and viscosity $\nu = 1$ at times $t = 0.01, 0.1$ and 1. The dotted line shows the initial velocity field.

The vorticity $\omega = \dfrac{1}{r}\dfrac{\partial}{\partial r}(rv) = \dfrac{C}{4\pi\nu t}e^{-r^2/[4\nu t]}$. Thus vorticity diffuses into the domain from the initial state in which it is 'concentrated' at the origin. However the total circulation remains unchanged during the evolution. Note that at long times ($\nu t \gg r^2$)

$$v(r,t) = \frac{C}{8\pi\nu t} r + \ldots, \tag{24}$$

which corresponds to rigid body rotation sufficiently close to the origin.