3 Dynamical similarity and the Reynolds Number

The velocity field $\mathbf{u}(\mathbf{x}, t)$ depends on the lengthscale of the object, L, the velocity scale, U, the fluid density, ρ and the dynamic viscosity, μ. It is insightful to recast the variables into dimensionless counterparts and to identify dimensionless groups that characterise the motion. We write

$$\mathbf{x} = L\mathbf{x}^*, \quad t = \frac{L}{U}t^*, \quad \mathbf{u} = U\mathbf{u}^* \quad \text{and} \quad p = \rho U^2 p^*, \tag{1}$$

where the variables denotes * are dimensionless. In these definition, we have chosen an inertial scaling for the pressure (ρU^2), as opposed to a viscous one ($\mu U/L$). The dimensionless equations are then

$$\nabla^* . \mathbf{u}^* = 0, \tag{2}$$

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* . \nabla^* \mathbf{u}^* = -\nabla^* p^* + \frac{1}{Re}\nabla^{*2}\mathbf{u}^*, \tag{3}$$

where the Reynolds number $Re = \rho UL/\mu$ is a dimensionless variable characterising the flow.

The Reynolds number may be interpreted as the following ratio,

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}} \sim \frac{\rho U^2/L}{\mu U/L^2}. \tag{4}$$

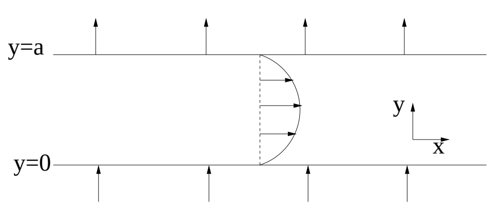
The magnitude of the Reynolds number determines the type of flow. Two important regimes emerge:

1. $Re \ll 1$. The equations become $0 = -\nabla p + \mu \nabla^2 \mathbf{u}$ to leading order. This is linear and instantaneous. This regime is applicable to small scales, slow or very viscous flows.

2. $Re \gg 1$. The effects of viscosity are negligible in many parts of the fluid domain, but may not be neglected throughout. Instead 'thin' structures develop near boundaries and these influence the overall motion. The motion is typically turbulent. This regime is applicable to large scales, fast and inviscid flows.

3.1 Channel flow with suction

We analyse two-dimensional flow in a horizontal channel between porous boundaries, such that there is an imposed flow, V, perpendicular to the channel axis.



The velocity field is of the form $\mathbf{u} = (u(y), V)$ and thus mass conservation is satisfied automatically. The x-component of the momentum equations is given by

$$\rho V \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, \tag{5}$$

while the y-component of the momentum equation is given by

$$0 = -\frac{\partial p}{\partial y}. \tag{6}$$

There are no slip boundary conditions at the edge of the channel $u(0) = u(a) = 0$.

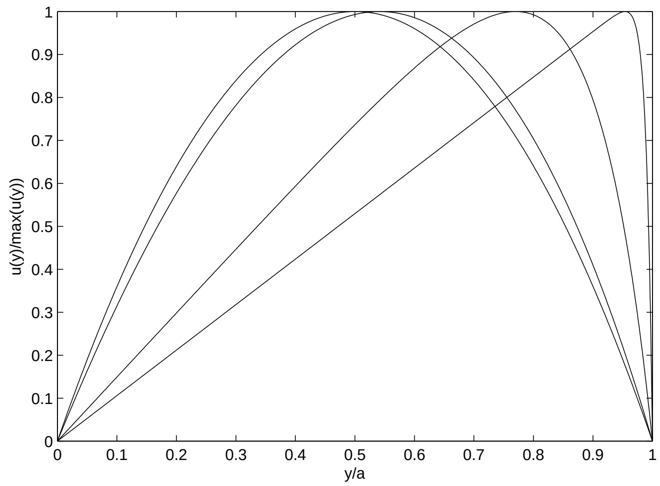


Figure 1: The axial velocity normalised by its maximum, $u(y)/\max(u(y))$, as a function of y/a for $Re = 0.01, 1, 10, 100$. (The flow profiles with higher Reynolds numbers are increasing asymmetric.)

From (6), we deduce that p is only a function of x, and then from (5) we deduce that $\partial p/\partial x$ must be constant as all the other terms are only functions of y. On writing $\partial p/\partial x = G$ (constant), we find that

$$\frac{u(y)}{V} = -\frac{Ga}{\rho V^2}\left(\eta - \frac{(1 - \mathrm{e}^{Re\eta})}{(1 - \mathrm{e}^{Re})}\right), \tag{7}$$

where $Re = Va/\nu$ and $\eta = y/a$. Here the Reynolds number measures the dimensionless strength of the suction velocity.

1. When $Re \ll 1$,

$$\frac{u(y)}{V} = -\frac{Ga}{\rho V^2}\frac{Re}{2}\eta(1 - \eta). \tag{8}$$

In this regime, the velocity approaches a parabolic profile and $u(y)$ becomes independent of V.

2. When $Re \gg 1$, if $Re(1 - \eta) \gg 1$ (i.e. away from the channel boundary), then

$$\frac{u(y)}{V} = -\frac{Ga}{\rho V^2}\eta. \tag{9}$$

However this linear profile does not hold as the boundary is approached. In particular, note that this approximate solution does not even satisfy the no-slip boundary condition at $y = a$. Thus the limit $Re \to \infty$ is a *singular limit*.