4 Flows with negligible inertia ($Re \ll 1$)

If the inertia of the fluid motion is negligible then the equations of motion reduce to a balance between the pressure gradient, viscous stresses and body forces. The governing equations are given by

$$\begin{aligned} \nabla . \mathbf{u} &= 0, && (1) \\ 0 &= -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}. && (2) \end{aligned}$$

These are known as the **Stokes equations**. This requires that $|\mu \nabla^2 \mathbf{u}| \gg |\rho \mathbf{u} . \nabla \mathbf{u}|$, which implies $1 \gg \rho U L/\mu$. Also $|\mu \nabla^2 \mathbf{u}| \gg |\rho \partial \mathbf{u}/\partial t|$, which implies that $1 \gg \rho L^2/[\mu T]$ and often $T \sim L/U$, which recovers the first regime.

The Stokes equations exhibit the following:

1. Instantaneous: the flow field responds immediately to changes in forcing and/or boundary conditions.

2. Linear: independent solutions can be superposed.

3. Reversible: the absence of inertia (acceleration) implies that a reversal in forcing ($\mathbf{F} \to -\mathbf{F}$) generates a reversal in velocity $\mathbf{u} \to -\mathbf{u}$.

4.1 Properties of Stokes flow

4.1.1 Uniqueness

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two different solutions of Stokes equations that satisfy the same boundary conditions. Thus

$$0 = -\nabla p_1 + \mu \nabla^2 \mathbf{u}_1 + \mathbf{F}, \qquad \nabla . \mathbf{u}_1 = 0, \qquad \mathbf{u}_1 = \mathbf{U} \quad \text{on } S, \tag{3}$$

and

$$0 = -\nabla p_2 + \mu \nabla^2 \mathbf{u}_2 + \mathbf{F}, \qquad \nabla . \mathbf{u}_1 = 0, \qquad \mathbf{u}_2 = \mathbf{U} \quad \text{on } S. \tag{4}$$

We define $\mathbf{V} = \mathbf{u}_1 - \mathbf{u}_2$ and $P = p_1 - p_2$ and find that

$$0 = \int_V \mathbf{V} . \left(-\nabla P + \mu \nabla^2 \mathbf{V} \right) \, \mathrm{d}V = \int_S -P V_i n_i + \mu V_i \frac{\partial V_i}{\partial x_j} n_j \, \mathrm{d}S - \int_V \mu |\nabla \mathbf{V}|^2 \, \mathrm{d}V. \tag{5}$$

But $\mathbf{V} = 0$ on S and thus the remaining volume integral vanishes. But this is positive semi-definite and can only vanish if $|\nabla \mathbf{V}| = 0$, which in turns implies that $\mathbf{V} = 0$ by virtue of the boundary conditions. Thus the two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ are equal.

4.1.2 Minimum dissipation

Let $\mathbf{u}$ be a solution of Stokes equations with no body forces and with given boundary conditions and let e_{ij}^u be the associated rate of strain tensor. Let $\mathbf{v}$ satisfy the same boundary conditions and $\nabla . \mathbf{v} = 0$ and let e_{ij}^v be the associated rate of strain tensor. Then

$$\int_V 2\mu e_{ij}^v e_{ij}^v \, \mathrm{d}V \geq \int_V 2\mu e_{ij}^u e_{ij}^u \, \mathrm{d}V. \tag{6}$$

This implies that Stokes flow has the minimum dissipation.

Proof: Let $\mathbf{v} = \mathbf{u} + \mathbf{w}$ so that

$$e^v_{ij} e^v_{ij} = e^u_{ij} e^u_{ij} + e^w_{ij} e^w_{ij} + 2 e^u_{ij} e^w_{ij}.$$

Then we can show that

$$\begin{aligned}
\int_V e^u_{ij} e^w_{ij} \, \mathrm{d}V &= \int_V e^u_{ij} \frac{\partial w_i}{\partial x_j} \, \mathrm{d}V && (7)\\
&= \int_S n_j e^u_{ij} w_i \, \mathrm{d}S + \int_V w_i \frac{1}{2} \frac{\partial^2 u_i}{\partial x_j^2} \, \mathrm{d}V \quad \text{(Integrating by parts)} && (8)\\
&= -\int_S \frac{1}{2} w_i p n_i \, \mathrm{d}S \quad \text{(Using governing equations for } u_i \text{ \& } w_i\text{)} && (9)\\
&= 0 \quad \text{(Using boundary condition for } w_i\text{)} && (10)
\end{aligned}$$

Thus

$$\int_V 2\mu e^v_{ij} e^v_{ij} \, \mathrm{d}V = \int_V 2\mu \left(e^u_{ij} e^u_{ij} + e^w_{ij} e^w_{ij} \right) \, \mathrm{d}V \geq \int_V 2\mu e^u_{ij} e^u_{ij} \, \mathrm{d}V. \tag{11}$$

This implies that Stokes flow produces the minimum dissipation.

4.1.3 Reciprocal theorem

Velocity fields $\mathbf{u}^1$ and $\mathbf{u}^2$ both satisfy Stokes equations subject to body forces $\mathbf{f}^1$ and $\mathbf{f}^2$, respectively, within a domain of volume V and bounding surface S with outward pointing unit normal $\mathbf{n}$. Then

$$I = \int_S u^1_i \sigma^2_{ij} n_j \, \mathrm{d}S + \int_V u^1_i f^2_i \, \mathrm{d}V - \int_S u^2_i \sigma^1_{ij} n_j \, \mathrm{d}S - \int_V u^2_i f^1_i \, \mathrm{d}V = 0. \tag{12}$$

Proof: Using divergence theorem and Stokes equations

$$\begin{aligned}
I &= \int_V \sigma^2_{ij} \nabla_j u^1_i - \sigma^1_{ij} \nabla_j u^2_i \, \mathrm{d}V && (13)\\
&= \int_V 2\mu e^2_{ij} \left(e^1_{ij} + \Omega^1_{ij} \right) - 2\mu e^1_{ij} \left(e^2_{ij} + \Omega^2_{ij} \right) \, \mathrm{d}V && (14)\\
&= 0. && (15)
\end{aligned}$$

4.2 Flow past a sphere

We consider uniform flow past a sphere and solve Stokes equations

$$0 = -\nabla p + \mu \nabla^2 \mathbf{u} \qquad \text{and} \qquad \nabla . \, \mathbf{u} = 0, \tag{16}$$

subject to $\mathbf{u} = 0$ on $r = a$ and $\mathbf{u} \to \mathbf{U}$ as $r \to \infty$. Since the governing equations are linear and the problem is axisymmetric, we seek the following representation of the solution

$$\mathbf{u} = \mathbf{U} f(r) + (\mathbf{U} . \, \mathbf{x}) \, \mathbf{x} g(r) \qquad \text{and} \qquad p = \mu \, (\mathbf{U} . \, \mathbf{x}) \, h(r), \tag{17}$$

where $r = |\mathbf{x}|$ and f, g and h are to be determined. Substituting into the expression of mass conservation gives

$$\nabla . \, \mathbf{u} = \mathbf{U} . \, \mathbf{x} \left(f' \frac{1}{r} + 4g + r g' \right) = 0. \tag{18}$$

The momentum equation becomes

$$-\nabla p + \mu \nabla^2 \mathbf{u} = \mu \mathbf{U} \left(-h + f'' + 2 f' \frac{1}{r} + 2g \right) + \mu \mathbf{U} . \, \mathbf{x} \left(g'' + 6 g' \frac{1}{r} - h' \frac{1}{r} \right) = 0. \tag{19}$$

Eliminating between (18) and (19), we find

$$r^2 g''' + 11 r g'' + 24 g' = 0, \tag{20}$$

which has solution

$$g(r) = A + B\frac{a^3}{r^3} + C\frac{a^5}{r^5}, \tag{21}$$

where A, B and C are constants to be determined. Also we find that

$$f = -2r^2 A + B\frac{a^3}{r} - C\frac{a^5}{3r^3} + D \quad \text{and} \quad h = -10A + 2B\frac{a^3}{r^3}, \tag{22}$$

where D is another constant. These functions (f, g, h) give the general solution for the velocity and pressure fields. We then apply the boundary conditions to find that

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \mathbf{x}\,(\mathbf{U}.\,\mathbf{x})\,\frac{3}{4a^2}\left(-\frac{a^3}{r^3} + \frac{a^5}{r^5}\right), \tag{23}$$

$$p = -\frac{3\mu a}{2}\frac{\mathbf{U}.\,\mathbf{x}}{r^3}. \tag{24}$$

The stress tensor may be evaluated directly from this expression. On the surface of the sphere, we find

$$\sigma_{ij} n_j = \frac{3\mu}{2a} U_i \qquad \text{on } r = a. \tag{25}$$

Then the drag force, F_i, exerted on the sphere is given by

$$F_i = \int_{r=a} \sigma_{ij} n_j \, \mathrm{d}S = 6\pi a \mu U_i. \tag{26}$$

Stokes settling velocity emerges from the steady balance between viscous drag and submerged gravitational weight. Thus for a spherical particle of density ρ_s submerged in fluid of density ρ_f settling with velocity w_s, the vertical force balance gives

$$\frac{4}{3}\pi a^3 (\rho_s - \rho_f)\, g = 6\pi\mu a w_s.$$

Thus we deduce that

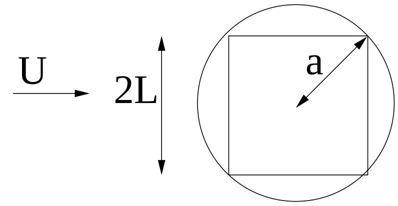
$$w_s = \frac{2(\rho_s - \rho_f) g a^2}{9\mu}. \tag{27}$$

4.2.1 Geometrical bounding

This is an application of the minimum dissipation theorem to address the question: what is the drag on a cube of side length $2L$?

The Stokes flow, $\mathbf{u}^S$, around the cube satisfies no-slip on its surface ($\mathbf{u}^S = 0$) , uniform flow in the far field ($\mathbf{u}^S \to \mathbf{U}$) and generates a drag force $\mathbf{F}$. Thus the dissipation is $\mathbf{F}.\,\mathbf{U}$.

We now consider the flow around a sphere of radius $a = \sqrt{3}L$ which circumscribes the cube. This generates a drag $6\pi\mu\sqrt{3}L\mathbf{U}$.

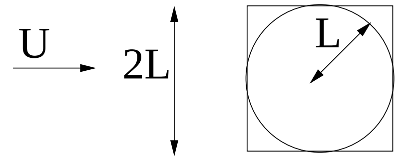


Suppose we have have a flow which is Stokes flow outside the circumscribing sphere, but vanishing inside, then it certainly satisfies no-slip on the cube surface, the far-field condition and it is divergent-free. The rate of dissipation for this flow is then $6\pi\mu\sqrt{3}L\mathbf{U}.\,\mathbf{U}$. But it is no longer a solution to Stokes equations. Thus the true Stokes flow around the cube has a smaller rate of dissipation and so

$$6\pi\mu\sqrt{3}L\mathbf{U}.\,\mathbf{U} \geq \mathbf{F}.\,\mathbf{U}. \tag{28}$$

Now consider an inscribing sphere of radius L. By a similar construction

$$\mathbf{F}.\mathbf{U} \geq 6\pi\mu L\mathbf{U}.\mathbf{U}. \tag{29}$$



Hence we have found both an upper and lower bound for the dissipation and thus for the settling velocity. This produces the general rule that the settling velocity of a non-spherical particle is bounded by those of the larger circumscribing sphere and the smaller inscribing sphere.

4.3 Flow past a spherical bubble

Provided the surface tension is sufficient strong to maintain the bubble in a spherical shape, we may use the solutions from §4.2 to calculate the flow field, pressure and drag on a bubble. We further assume that the motion of gas in the bubble can be neglected so that the boundary conditions on the velocity field are $\mathbf{u} \to \mathbf{U}$ as $r \to \infty$, while on the surface of the bubble

$$\mathbf{u}.\mathbf{n} = 0 \quad \text{at } r = a \qquad \text{(No normal velocity)} \tag{30}$$

$$t_i\sigma_{ij}n_j = 0 \quad \text{at } r = a \qquad \text{(No tangential shear stress)}, \tag{31}$$

where t_i and n_i are respectively tangential and normal vectors to the surface $r = a$. Enforcing these boudnary conditions, we find that

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{a}{2r}\right) + \mathbf{x}\frac{(\mathbf{U}.\mathbf{x})}{a^2}\left(-\frac{a^3}{2r^3}\right), \tag{32}$$

$$p = -\frac{\mu a\mathbf{U}.\mathbf{x}}{r^3}. \tag{33}$$

The surface stress can then be shown to be

$$\sigma_{ij}n_j = \frac{3\mu}{a^3}\mathbf{U}.\mathbf{x}x_i, \qquad \text{on } r = a \tag{34}$$

Since $\displaystyle\int_{r=a} x_i x_j \mathrm{d}S = \frac{4}{3}\pi a^4\delta_{ij}$. This implies that the drag force, F_i, is

$$F_i = 4\pi\mu a U_i, \tag{35}$$

and so the rise speed of a bubble of gas of density $\rho_g < \rho_f$ is $(\rho_f - \rho_g)ga^2/(3\mu)$.

4.4 Two-dimensional flow in corners

When the flow is two-dimensional we may introduce a streamfunction $\mathbf{u} = \nabla \wedge \psi(x, y)\hat{\mathbf{z}}$. In such situations the vorticity is $\boldsymbol{\omega} = -\nabla^2\psi\hat{\mathbf{z}}$. We take the curl of the Stokes equations to find that

$$0 = \nabla^2\boldsymbol{\omega} = \nabla^4\psi\hat{\mathbf{z}}. \tag{36}$$

Thus the streamfunction satisfies the *bi-harmonic equation.*

We look for separable solutions in polars coordinates $\psi(r, \theta) = r^\lambda f(\theta)$, which leads to

$$\nabla^4\psi = r^{\lambda-4}\left(\frac{\mathrm{d}^4 f}{\mathrm{d}\theta^4} + ((\lambda-2)^2 + \lambda^2)\frac{\mathrm{d}^2 f}{\mathrm{d}\theta^2} + (\lambda-2)^2\lambda^2 f\right) = 0. \tag{37}$$

Thus the general solution is

$$f(\theta) = A\cos\lambda\theta + B\sin\lambda\theta + C\cos(\lambda-2)\theta + D\sin(\lambda-2)\theta, \tag{38}$$

where A, B, C and D are constants. If $\lambda = 0, 1, 2$ then the solution is different and given by

$$f = A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta \qquad \text{if } \lambda = 1, \tag{39}$$

$$f = A\cos 2\theta + B\sin 2\theta + C + D\theta \qquad \text{if } \lambda = 0, 2 \tag{40}$$

4.4.1 Scraper

Viscous fluid is dragged over a rigid boundary by a scraper inclined at angle α to the boundary. In a frame moving with the scraper, the boundary conditions are

$$\mathbf{u} = -U\hat{\mathbf{r}} \qquad \text{on } \theta = 0 \text{ and} \qquad \mathbf{u} = 0 \qquad \text{on } \theta = \alpha. \tag{41}$$

In terms of the streamfunction $\psi(r,\theta)$ these conditions correspond to

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U \text{ and } \frac{\partial \psi}{\partial r} = 0 \qquad \theta = 0, \tag{42}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \text{ and } \frac{\partial \psi}{\partial r} = 0 \qquad \theta = \alpha. \tag{43}$$

Using the general solution (39) and applying the boundary conditions, we find that

$$\psi(r,\theta) = Ur\frac{(\theta \sin\alpha \sin(\alpha-\theta) - \alpha(\alpha-\theta)\sin\theta)}{\alpha^2 - \sin^2\alpha}. \tag{44}$$

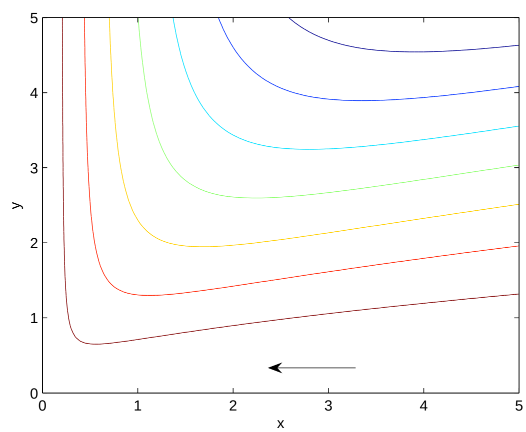
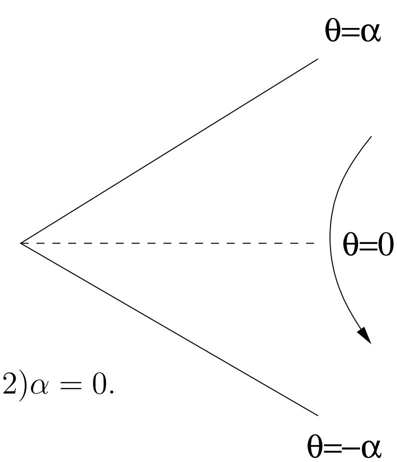


Figure 1: The streamlines in a frame moving with the fluid for $\alpha = \pi/2$ and $U = 1$.

4.4.2 Corner flows: *Moffatt eddies*



The flow in a corner of angle 2α is forced by motion away from the corner.

We seek a solution that is symmetric in the line $\theta = 0$ and write the streamfunction as

$$\psi = r^\lambda(A\cos\lambda\theta + C\cos(\lambda-2)\theta). \tag{45}$$

The no-slip boundary condition at $\theta = \pm\alpha$ requires that $f(\alpha) = 0$ and $\mathrm{d}f/\mathrm{d}\theta(\alpha) = 0$ and so

$$A\cos\lambda\alpha + C\cos(\lambda-2)\alpha = 0 \qquad \text{and} \qquad A\lambda\sin\lambda\alpha + C(\lambda-2)\sin(\lambda-2)\alpha = 0. \tag{46}$$

Thus for a non-trivial solution, we require

$$(\lambda-2)\tan(\lambda-2)\alpha = \lambda\tan\lambda\alpha. \tag{47}$$

But are there real solutions for λ to (47)? This is most easily assessed by writing $\lambda = 1 + \beta$ and then (47) becomes

$$\frac{\sin \Lambda}{\Lambda} = -\frac{\sin 2\alpha}{2\alpha}, \tag{48}$$

where $\Lambda = 2\beta\alpha$. The function $(\sin \Lambda)/\Lambda$ has a minimum value of -0.217 and so we require that $(\sin 2\alpha)/(2\alpha) < 0.217$. This implies that there are real solutions provided $\alpha > \alpha_* = 73.2°$.

For $\alpha < \alpha_*$, there are complex solutions for λ. Writing $\lambda = a + ib$ and $f_r + \mathrm{i} f_i$, this gives the streamfunction as

$$\psi = r^a \left(f_r \cos(b \ln r) - f_i \sin(b \ln r) \right). \tag{49}$$

This means that for fixed angle, θ, the sign of the streamfunction oscillates as r is varied. Thus the motion has eddies with successive different directions of circulation.

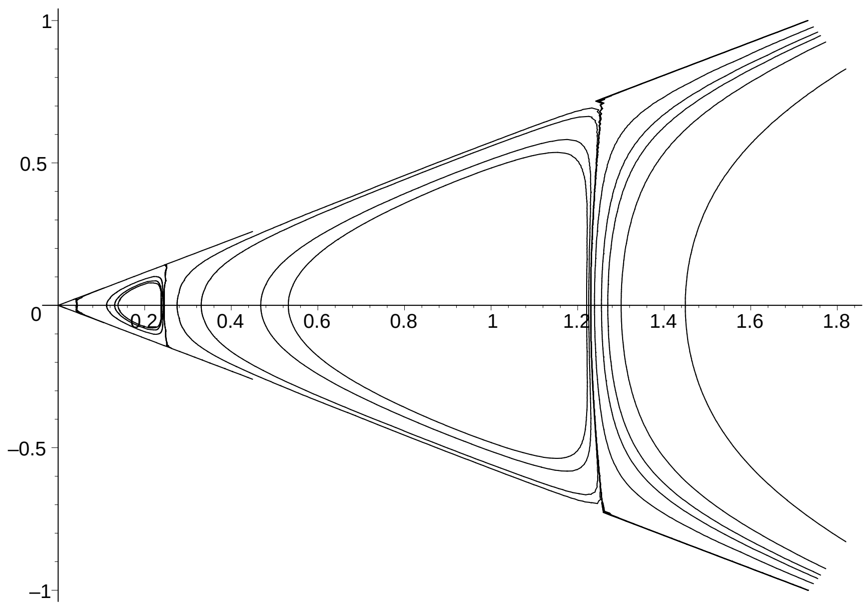


Figure 2: The streamlines for flow in a corner ($\alpha = \pi/6$). Note the counter-rotating Moffatt eddies.

4.5 Lubrication theory: *Flows in thin films*

Lubrication flows are characterised by one spatial dimension being much smaller than another. For example, consider the flow between two nearby solid surfaces or spreading of a droplet.

To develop the leading order description of the flow, we assume that the extensive direction is parallel with the x-axis, while the relatively thin direction is parallel with the z-axis. We denote the lengthscales in each of these directions by L and h and we require that $h/L \ll 1$.

First, incompressibility demands that $\nabla . \mathbf{u} = 0$ and for a 2-D flow with $\mathbf{u} = (u, w)$, this implies that the two velocity scales are related by $W \sim hU/L$.

Next we assess the magnitude of terms in the x-momentum equations,

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{50}$$

Given that $h/L \ll 1$, we deduce that $|\partial^2 u/\partial x^2| \ll |\partial^2 u/\partial z^2|$. Further $|\partial^2 u/\partial z^2| \gg |\rho Du/Dt|$ if $(h/L)^2 Re \ll 1$, where the Reynolds number $Re = \rho UL/\mu$. The only residual term in (50) is the streamwise pressure gradient and this is required to drive the flow. Thus the scale of the pressure $P \sim L\mu U/h^2$.

Using these scalings, we may now assess the magnitude of terms in the z-momentum equation,

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

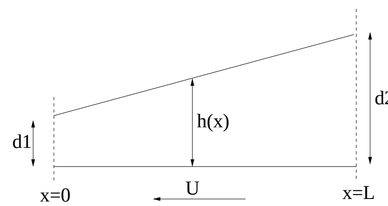
We find that $\rho Dw/Dt \sim \rho h U^2/L^2$, $\partial p/dpz \sim \mu L U/h^3$ and $\mu\partial^2 w/\partial z^2 \sim \mu U/(hL)$. Thus we find that the pressure gradient $\partial p/\partial z$ is much larger than all of the other terms.

Thus the leading order lubrication equations are

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2}, \tag{51}$$

$$0 = -\frac{\partial p}{\partial z}. \tag{52}$$

4.5.1 Thrust bearing



We consider motion of a thin layer of fluid between a stationary planar bearing above a moving horizontal surface. The thickness of the gap between the bearing and the surface is given by

$$h(x) = d_1 + \frac{d_2 - d_1}{L}x, \tag{53}$$

where $|h/L| \ll 1$, so that the motion can be analysed in the lubrication regime.

The lubrication equations (51)-(52) imply the pressure $p \equiv p(x)$ and that the velocity field is given by

$$u = -\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}z(h-z) - U\frac{(h-z)}{h}, \tag{54}$$

which is constructed to satisfy no slip conditions $u(h) = 0$ and $u(0) = -U$. Then the volume flux of fluid per unit width is

$$Q = \int_0^h u\,\mathrm{d}z = -\frac{Uh}{2} - \frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{55}$$

Since this flow is in a steady-state, Q must be constant. Integrating to find the pressure and equating the pressure at $x = 0$ and $x = L$ then yields

$$0 = \int_0^L \frac{Q + \frac{1}{2}Uh}{h^3}\,\mathrm{d}x \quad \text{which gives} \quad Q = -\frac{Ud_1 d_2}{d_1 + d_2}. \tag{56}$$

It is noteworthy that the pressure reaches a maximum when $Q + Uh/2 = 0$ and this occurs at $x = Ld_1/(d_1 + d_2)$. The shear stress at the wall, τ, is given by

$$\tau = \mu\frac{\partial u}{\partial z} = \mu\frac{U}{h}\left(4 + 6\frac{Q}{hU}\right). \tag{57}$$

and so the shear stress vanishes at $h = -3Q/(2U) = 3d_1 d_2/(2(d_1 + d_2))$.

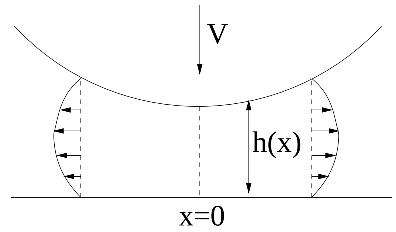
The total normal and tangential forces on the lower plate per unit width, denoted by N and T, respectively, are given by

$$N = \int_0^L \sigma_{zz}\,\mathrm{d}x = \frac{6\mu L^2 U}{(d_2 - d_1)^2}\left(\ln\left(\frac{d_2}{d_1}\right) - \frac{2(d_2 - d_1)}{d_2 + d_1}\right) \tag{58}$$

$$T = \int_0^L \sigma_{xz}\,\mathrm{d}x = \frac{2\mu L U}{d_2 - d_1}\left(2\ln\left(\frac{d_2}{d_1}\right) - \frac{3(d_2 - d_1)}{d_1 + d_2}\right). \tag{59}$$

Thus $T/N \sim (d_2 - d_1)/L \ll 1$. This means that this is a low friction bearing. The fluid 'lubricates' the motion.

4.5.2 Cylinder approaching a wall



A cylinder of radius a approaches a wall with speed V. When the centreline gap d is small ($d \ll a$), the motion may be treated using lubrication theory.

The gap size, $h(x)$, is given by

$$h(x) = a + d - (a^2 - x^2)^{1/2} = d\left(1 + \frac{x^2}{2ad} + \dots\right). \qquad (60)$$

We truncate this expansion at $O(x^2)$ and analyse the fluid motion on vertical lengthscales d and horizontal lengthscales $\sqrt{ad}$. The lubrication equations (51)-(52) then imply that $p \equiv p(x)$ and

$$u = -\frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}(h - z)z, \qquad (61)$$

where the velocity field is constructed to satisfy no-slip conditions: $u(0) = 0$ and $u(h) = 0$. The volume flux per unit width, Q, is given by

$$Q = \int_0^h u \,\mathrm{d}z = -\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}. \qquad (62)$$

Mass conservation gives

$$\frac{\partial}{\partial x}\int_0^h u \,\mathrm{d}z - u(h)\frac{\partial h}{\partial x} + w(h) = 0, \quad \text{which implies} \quad \frac{\partial Q}{\partial x} = V \quad \text{and so} \quad Q = Vx. \qquad (63)$$

Substituting into (62) allows the pressure field to be deduced

$$p - p_\infty = \frac{6\mu V a}{d^2\left(1 + \frac{x^2}{2ad}\right)^2}, \qquad (64)$$

where p_∞ is the far-field pressure. Thus the normal force on the cylinder per unit width is given by

$$F = \int_{-\infty}^{\infty}(p - p_\infty)\,\mathrm{d}x = 3\sqrt{2}\pi\mu V\left(\frac{a}{d}\right)^{3/2}. \qquad (65)$$

We now identify that the approach velocity is $V = -\mathrm{d}d/\mathrm{d}t$. Thus the force balance implies

$$\pi\Delta\rho g a^2 = -3\sqrt{2}\pi\mu\left(\frac{a}{d}\right)^{3/2}\frac{\mathrm{d}d}{\mathrm{d}t},$$

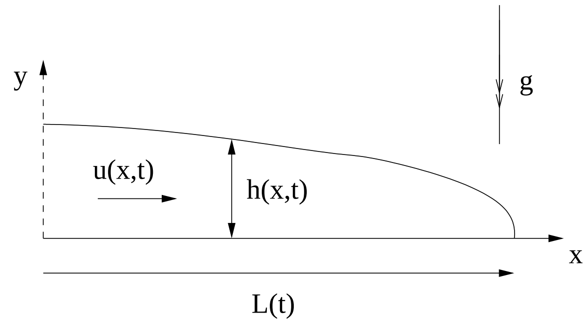
which may be integrated to show that

$$d(t) = \frac{a}{\left(c + \frac{\Delta\rho a g t}{6\sqrt{2}\mu}\right)^2}, \qquad (66)$$

where c is a constant. Thus $d \sim t^{-2}$ as $t \to \infty$.

4.5.3 Gravitationally spreading drop

We analyse a low aspect ratio drop ($h/L \ll 1$) spreading gravitationally over a horizontal surface. The lubrication equations now include gravity so that they are given by



$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \quad (67)$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \quad (68)$$

The pressure is therefore in hydrostatic balance and is given by

$$p = p_0 + \rho g(h - z), \quad (69)$$

where p_0 is atmospheric pressure at the free-surface. Gradients of this field drive the horizontal motion. The horizontal velocity is then given by

$$u = -\frac{\rho g}{2\mu}\frac{\partial h}{\partial x} z(2h - z). \quad (70)$$

This satisfies conditions on no-slip at the underlying plane ($u(0) = 0$) and no stress at the free surface ($\partial u/\partial z(h) = 0$). The volume flux per unit width is

$$Q = \int_0^h u \, \mathrm{d}z = -\frac{\rho g h^3}{3\mu}\frac{\partial h}{\partial x}. \quad (71)$$

From mass conservation we deduce that

$$0 = \frac{\partial}{\partial x}\int_0^h u \, \mathrm{d}z - u(h)\frac{\partial h}{\partial x} + w(h) = \frac{\partial Q}{\partial x} + \frac{\partial h}{\partial t}, \quad (72)$$

using the kinematic condition on the free surface ($D(z - h)/Dt = 0$ at $z = h$). Thus we derive the following nonlinear diffusion equation governing the evolution of $h(x, t)$,

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\rho g h^3}{3\mu}\frac{\partial h}{\partial x}\right). \quad (73)$$

This is to be solved subject to the conditions: (i) there is no flow at $x = 0$, $\partial h/\partial x = 0$; (ii) the height vanishes at the front $h(L(t), t) = 0$; and (iii) the volume per unit width of material in the current is V

$$\int_0^{L(t)} h \, \mathrm{d}x = V. \quad (74)$$

We seek a similarity solution to (73) of the form $h(x, t) = t^\beta H(x/t^\alpha)$ and $L = Ct^\alpha$, where the constant α, β and C and the function F are to be determined.

From (73) we find that $2\alpha - 3\beta = 1$ and from (74) that $\beta + \alpha = 1$. Thus $\alpha = -\beta = 1/5$ and that the solution is given by

$$h = V\left(\frac{9\mu}{10\rho g V^3 t}\right)^{1/5} \Gamma^{-2/5}\left(1 - \frac{x^2}{L(t)^2}\right)^{1/3} \quad (75)$$

$$L(t) = \Gamma^{-3/5}\left(\frac{10\rho g V^3 t}{9\mu}\right)^{1/5} \quad (76)$$

where $\eta = x/L(t)$ and $\Gamma = \int_0^1 (1 - \eta^2)^{1/3} \, \mathrm{d}\eta = 0.8413$.

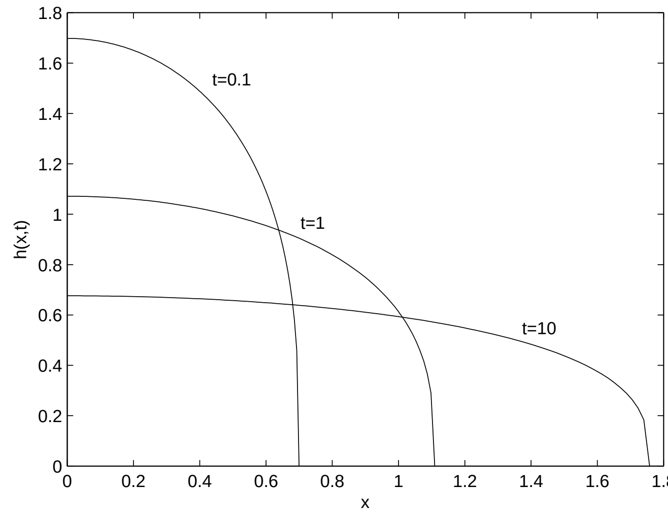


Figure 3: The height of the spreading droplet as a function of distance for $t = 0.1, 1, 10$. In this plot $V = 1$ and $10\rho g V^3/(9\mu) = 1$.