5 Flows at high Reynolds numbers

5.1 The Prandtl and Euler limits

In dimensionless variables the Navier-Stokes equations are given by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}, \tag{1}$$

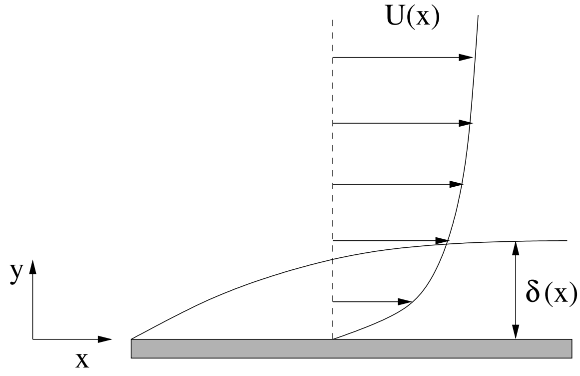
where Re denotes the Reynolds number. When the Reynolds number is large ($Re \gg 1$), it is tempting to approximate the Navier-Stokes equation by neglecting the term that corresponds to the divergence of the viscous stresses, namely the final term on the righthand side of (1). This leads to the Euler equation for inviscid fluids:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\nabla p. \tag{2}$$

However the limit $Re \to \infty$ is singular and the total neglect of viscous stresses implies that energy is not dissipated by the flow and that a no-slip condition at boundaries can not be enforced. Mathematically we see that this limit is singular because the terms involving the highest derivatives are neglected. Instead the Prandtl limit recognises that when $Re \gg 1$, there is a need to consider variations in the velocity field close to boundaries, interfaces and other non-uniformities, on much shorter lengthscales so that the viscous effects are non-negligible. This implies that the flow contains 'boundary layer' structures over which the velocity field may vary very substantially and within which viscous processes play an important role. Re-assessing the lengthscale of the flow and identifying a much shorter length of variation then promotes the magnitude of the viscous term in (1). Establishing a solution then comprises calculating the solution within the boundary layer and ensuring that it matches onto a solution away from the boundary. This asymptotic technique for calculating the flow is known as 'matched asymptotic expansions'.

5.2 Boundary layer equations

We analyse a two-dimensional flow over a rigid stationary boundary with velocity field $\mathbf{u} = (u, v)$, such that far from the boundary the flow field is given by $u = U(x)$. The boundary layer is assumed to be of thickness $\delta(x)$, while the streamwise lengthscale over which the velocity varies is denoted by L. The boundary layer equations are derived in the regime $\delta/L \ll 1$, corresponding to a relatively thin boundary layer.



The flow is incompressible, so this leads to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

If we estimate the size of the terms in this expression (3), we find $u/\delta \sim v/L$. Thus $v/u \sim \delta/L \ll 1$, confirming that the flow within the boundary layer is essentially parallel with the underlying boundary. The x-component of the Navier-Stokes equation and the estimates of the various terms are given by

$$\rho(\frac{\partial u}{\partial t} \qquad +u\frac{\partial u}{\partial x} \qquad +v\frac{\partial u}{\partial y}) \qquad = -\frac{\partial p}{\partial x} \qquad +\mu\nabla^2 u. \tag{4}$$

$$\rho(\frac{u}{T}, \qquad \frac{u^2}{L}, \qquad \frac{vu}{\delta}) \qquad \frac{p}{L}, \qquad \mu\left(\frac{u}{L^2}, \frac{u}{\delta^2}\right) \tag{5}$$

On the basis that $\delta/L \ll 1$, we deduce that $\nabla^2 u \sim \partial^2 u/\partial y^2$ to leading order. Furthermore the timescale T is determined by advection so $T \sim L/u$ and thus since $v \sim \delta u/L$, all of the terms on the lefthand side of (2) are of order $\rho u^2/L$. The pressure is then chosen to match this scale and so $p \sim \rho u^2$; in other words, it is set by inertial flow processes. Finally, within the boundary layer, the viscous term must balance the inertial terms and so this demands

$$\frac{\rho u^2}{L} \sim \frac{\mu u}{\delta^2}, \qquad \text{which implies} \qquad \frac{\delta}{L} \sim \left(\frac{\mu}{\rho U L}\right)^{1/2} = Re^{-1/2}. \tag{6}$$

This determines the thickness of the boundary layer and since $Re \gg 1$, it confirms that $\delta/L \ll 1$.

Next turning to the y-component of the momentum equation, we can estimate the magnitude of the terms.

$$\rho\frac{Dv}{Dt} \sim \frac{\rho u^2 \delta}{L^2}, \qquad -\frac{\partial p}{\partial y} \sim \frac{\rho u^2}{\delta}, \qquad \text{and} \qquad \mu\nabla^2 v \sim \frac{\mu u}{L\delta}. \tag{7}$$

This implies that the pressure gradient is much larger than all of the other terms.

The boundary layer equations are therefore

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{8}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \tag{9}$$

$$0 = -\frac{\partial p}{\partial y}. \tag{10}$$

The boundary conditions are those of no-slip at the boundary $u = v = 0$ on $y = 0$, while matching to the inviscid fluid away from the boundary demands $u \to U(x,t)$ as $y \to \infty$.

The boundary layer equations imply that the pressure is independent of y and so it is determined by its value outside of the boundary layer, where viscous stresses are negligible and so

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) = -\frac{\partial p}{\partial x}. \tag{11}$$

5.3 Flow over a flat plate: *Blasius boundary layer*

In this flow, away from the boundary, the outer streamwise velocity and pressure are constants. This means there is no lengthscale for the streamwise variation other than the distance from the start of the boundary, x, and so the thickness of the boundary is given by

$$\delta(x) = \left(\frac{\nu x}{U}\right)^{1/2}, \tag{12}$$

while the Reynolds number $Re = Ux/\nu$. The steady boundary layer equations are tackled by introducing a streamfunction so that mass conservation is automatically enforced. Thus

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}. \tag{13}$$

A similarity solution is sought for the streamfunction by writing it in the form

$$\psi = U\delta(x) f\left(\frac{y}{\delta}\right), \tag{14}$$

where f is to be determined. For the boundary layer equations, we need

$$u = Uf', \qquad \frac{\partial u}{\partial y} = \frac{U}{\delta} f'' \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{U}{\delta^2} f'''. \tag{15}$$

Also writing the similarity solution $\xi = y/\delta(x)$,

$$v = \frac{U\delta}{2x}\left(\xi f' - f\right) \qquad \text{and} \qquad \frac{\partial u}{\partial x} = -\frac{U\xi f''}{2x}. \tag{16}$$

Thus the boundary layer (9) gives

$$-\frac{1}{2} f f'' = f'''. \tag{17}$$

The boundary conditions are: (i) no tangential velocity on the boundary $u = 0$, which implies $f'(0) = 0$; (ii) no normal velocity on the boundary $v = 0$, which implies $f(0) = 0$; and (iii) matching the velocity far from the plane $u \to U$ as $y \to \infty$, which implies $f' \to 1$ as $\xi \to \infty$. This problem is solved using numerical techniques.

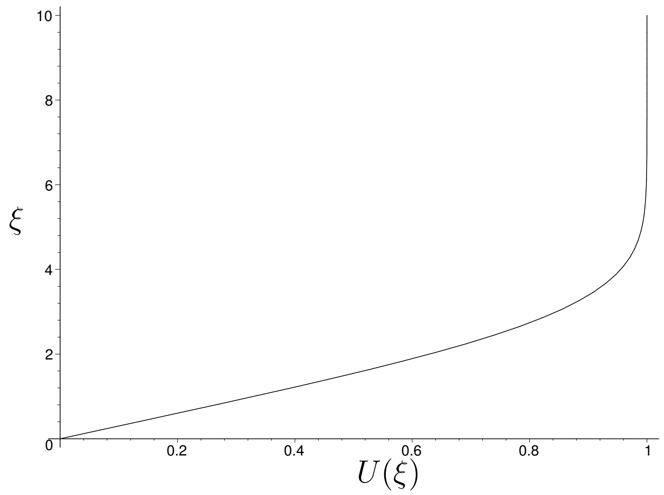


Figure 1: The velocity profile in the Blasius boundary layer

Two important quantities emerge. The shear stress on the boundary, τ, is given by

$$\tau = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0.332\rho U^2 \left(\frac{\nu}{Ux}\right)^{1/2}. \tag{18}$$

Also the boundary layer thickness is sometimes measured by

$$\int_0^\infty \left(1 - \frac{u}{U}\right) \,\mathrm{d}y = 1.72 \left(\frac{x\nu}{U}\right)^{1/2}. \tag{19}$$

5.4 The effects of acceleration and deceleration on a boundary layer: *Falkner & Scan*

Suppose the steady external flow is $U(x) = cx^m$ and there is a thin boundary along the x-axis in $x > 0$. The outer flow then corresponds to the streamfunction, ψ, which in polar coordinates is given by

$$\psi = \frac{c}{m+1} r^{m+1} \sin((m+1)\theta). \tag{20}$$

The outer flow field then corresponds to flow around a corner of exterior angle $\pi/(m+1)$. When $m > 0$ the outer flow is accelerating, whereas when $m < 0$ the flow is decelerating.

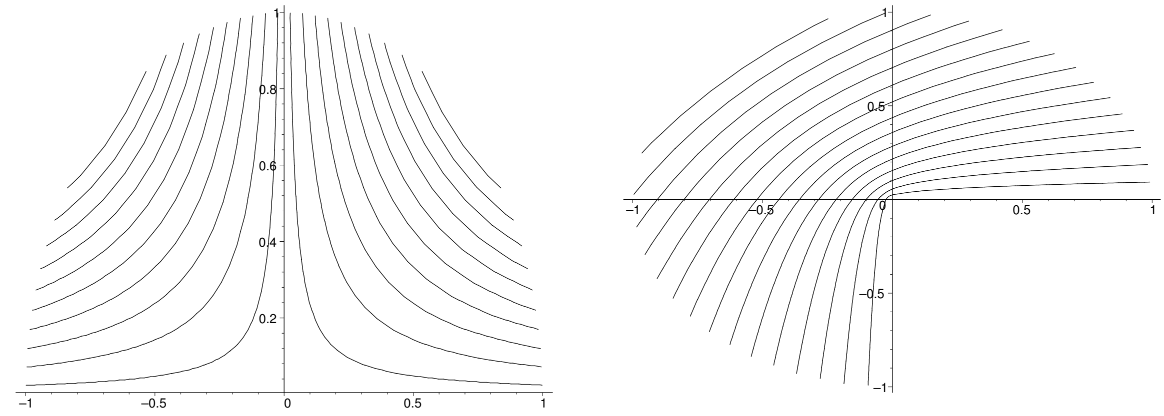


Figure 2: The streamlines corresponding to (i) Outer flow with $\psi = \frac{c}{2}r^2\sin(2\theta)$ $(m = 1)$; and (ii) Outer flow with $\psi = \frac{3c}{2}r^{2/3}\sin(2\theta/3)$ $(m = -1/3)$.

We may estimate the boundary layer thickness by balancing the inertial and viscous terms, $\rho U^2/x \sim \mu U/\delta^2$ and thus the boundary layer thickness is given by

$$\delta = \left(\frac{\nu x}{c x^m}\right)^{1/2}. \tag{21}$$

We seek a similarity solution for the steady streamfunction modelling the boundary layer flow,

$$\psi = U(x)\delta(x) f\left(\frac{y}{\delta(x)}\right) = c x^m \left(\frac{\nu x}{c x^m}\right)^{1/2} f(\xi), \tag{22}$$

where $\xi = y/\delta(x)$. For the boundary layer equations, we need the following derivatives of the velocity field

$$u = U(x) f'(\xi) \qquad \frac{\partial u}{\partial y} = \frac{U(x)}{\delta} f'' \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{U(x)}{\delta^2} f'''. \tag{23}$$

Also we need the following

$$v = -U'\delta f - U\delta' f + U\xi\delta' f' \qquad \text{and} \qquad \frac{\partial u}{\partial x} = U' f' - \frac{U\xi\delta'}{\delta} f''. \tag{24}$$

Thus on substitution into the boundary layer equation, we find that

$$f''' + m = m f'^2 - \frac{1}{2}(m+1) f f''. \tag{25}$$

This is subject to boundary conditions: (i) no tangential velocity on the boundary $u = 0$, which implies $f'(0) = 0$; (ii) no normal velocity on the boundary $v = 0$, which implies $f(0) = 0$; and (iii) matching the velocity far from the plane $u \to U(x)$ as $y \to \infty$, which implies $f' \to 1$ as $\xi \to \infty$. This problem is solved using numerical techniques.

- Solutions may be found for accelerating flows $m > 0$.
- When $-0.0904 < m < 0$, there are multiple solutions, with one (unphysical) type involving reverse flow.

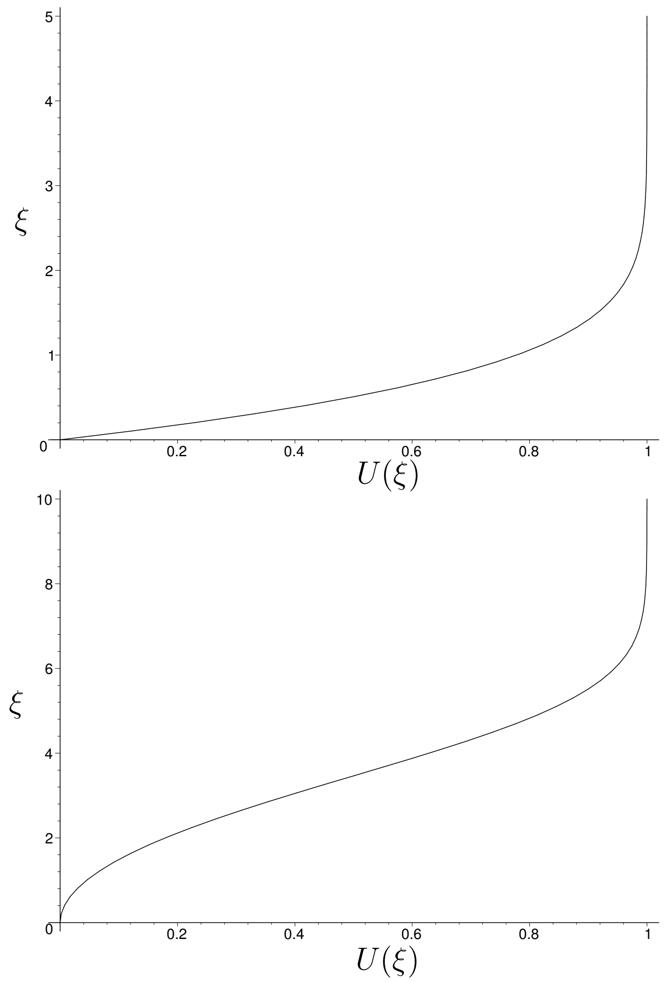
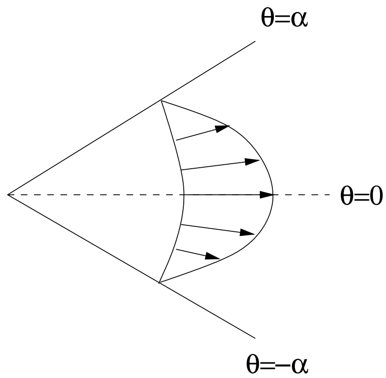


Figure 3: The velocity profile for (i) Stagnation flow $m = 1$; and (ii) Outer flow with $m = -0.0904$, for which the boundary shear stress vanishes.

- When $m = -0.0904$ the shear stress vanishes on the plate boundary, $\dfrac{\partial u}{\partial y} = 0$.
- For $m < -0.0904$, no solutions exist.

The overall conclusion is that boundary layers can not be found when there is significant deceleration in the outer flow. This arises because the deceleration leads to the expulsion of fluid away from the boundary to conserve mass. But this also transports horizontal momentum away from the boundary and it may not be balanced by viscous diffusion of momentum towards the boundary. Instead the boundary layer detaches and the flow 'separates'.

5.5 Flow in a diverging channel: *Jeffrey-Hamel flow*



This two-dimensional motion occurs in a wedge of angle 2α, driven by a sustained flux of fluid at the apex of the wedge. The radial motion therefore progressively slows to conserve mass. This is another example of flow where the motion distant from the boundary is decelerating. We therefore expect not to be able to find a boundary layer solution in all situations.

We seek a purely radially velocity field of the form

$$\mathbf{u} = \frac{F(\theta)}{r}\hat{\mathbf{r}}, \qquad \text{where} \qquad F(0) = F_0. \tag{26}$$

This form of velocity field automatically satisfies mass conservation, while between the momentum equations in the angular and radial directions we find that

$$\nu F''' + 2F'F + 4\nu F' = 0. \tag{27}$$

We substitute $F = F_0 f(\theta/\alpha)$ and thus

$$f''' + 2\alpha Re f f' + 4\alpha^2 f' = 0, \tag{28}$$

subject to $f(0) = 1$ and $f(1) = f(-1) = 0$. The Reynolds number $Re = \alpha F_0/\nu$ and we assume that the flow is symmetric so that $f'(0) = 0$. Integrating the governing equation twice, we find that

$$\frac{1}{2}f'^2 + \frac{1}{3}\alpha Re f^3 + 2\alpha^2 f^2 + c_1 f + c_2 = 0. \tag{29}$$

Using the boundary conditions, we find that $c_2 = -\frac{1}{2}f'(1)^2 < 0$ and that $\frac{1}{3}\alpha Re + 2\alpha^2 + c_1 + c_2 = 0$. Further integrating (2) and provided there is only one turning point in the flow where f' vanishes, we find

$$1 = \int_0^1 \mathrm{d}\eta = \int_0^1 \frac{1}{\left(\frac{2}{3}\alpha Re(f^2+f) + 4\alpha^2 f - 2c_2)^{1/2}(1-f)^{1/2}\right)}\mathrm{d}f. \tag{30}$$

Since $c_2 < 0$ this means that we can deduce

$$\int_0^1 \frac{1}{(f(1-f^2))^{1/2}}\,\mathrm{d}f \geq \left(\frac{2}{3}\alpha Re\right)^{1/2}. \tag{31}$$

Thus $\alpha Re \leq 10.31$. If the Reynolds number exceeds this value then some of the assumptions about the form of the velocity breaks down. In particular, we find that there are regions of reversed flow. In fact these steady states with reversed flow are not seen in experiments; instead the flow is unsteady.

This is another manifestation of the fact that boundary layers can not tolerate much deceleration at high Reynolds numbers.

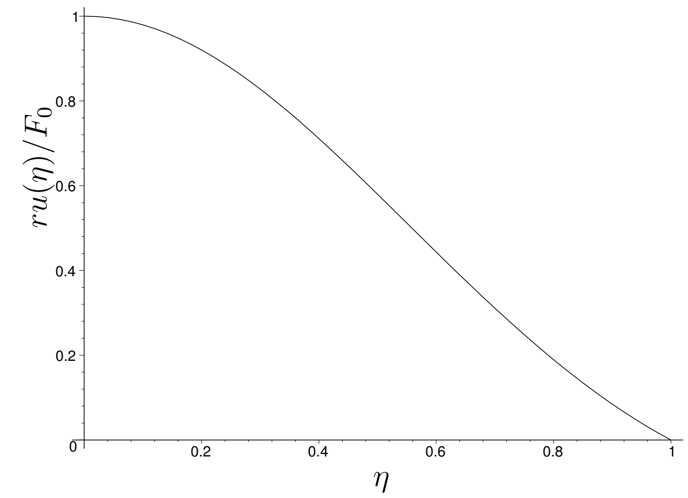
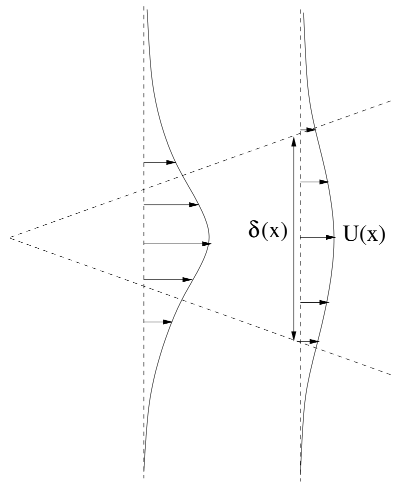


Figure 4: The radial velocity profile $ru/F_0 = f(\eta)$ for $Re = 5$ and $\alpha = \pi/4$.

5.6 Two-dimensional momentum jet

At high Reynolds number, fluid emerging from a source remains in a thin sheet and even though it is not adjacent to a boundary, we may use the boundary layer equations to model its evolution because it is relatively thin. The problem, though, is how to choose the scale of the thickness of the flow because it is not imposed by an external flow. The momentum flux is defined by

$$F = \int_{-\infty}^{\infty} \rho u^2 \, \mathrm{d}y. \tag{32}$$



We may show that this remains constant:

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \int_{-\infty}^{\infty} 2\rho u \frac{\partial u}{\partial x} \, \mathrm{d}y &(33)\\
&= 2\rho \int_{-\infty}^{\infty} \nu \frac{\partial^2 u}{\partial y^2} - v \frac{\partial u}{\partial y} \, \mathrm{d}y &(34)\\
&= 2\rho \int_{-\infty}^{\infty} \nu \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial y}(uv) + u \frac{\partial v}{\partial y} \, \mathrm{d}y &(35)\\
&= 2\rho \int_{-\infty}^{\infty} \nu \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial y}(uv) - u \frac{\partial u}{\partial x} \, \mathrm{d}y &(36)\\
&= 2\rho \left[\nu \frac{\partial u}{\partial y} - uv \right]_{-\infty}^{\infty} - \frac{\mathrm{d}F}{\mathrm{d}x}. &(37)
\end{aligned}$$

Thus using the boundary conditions as $|x| \to \infty$, we deduce that $\mathrm{d}F/\mathrm{d}x = 0$ and so $F = F_0$ (constant). This could have been anticipated on physical grounds because there is no means for loss of momentum flux. The momentum flux may then be used to set the scale of the centreline velocity of the jet, $U(x)$, and its width, $\delta(x)$, on the grounds that the distinguished scaling for the width balances inertial and viscous terms. Thus we find

$$U \sim \left(\frac{F_0^2}{\rho^2 \nu x} \right) \quad \text{and} \quad \delta \sim \left(\frac{\nu^2 x^2 \rho}{F_0} \right). \tag{38}$$

We now introduce a streamfunction in similarity form

$$\psi = U(x)\delta(x)f\left(\frac{y}{\delta}\right) = \left(\frac{F_0\nu x}{\rho}\right)^{1/3} f\left(\frac{y}{\delta}\right). \tag{39}$$

On substitution to the boundary layer equations, we find that

$$f''' + \frac{1}{3}f'^2 + \frac{1}{3}ff'' = 0, \tag{40}$$

where $\xi = y/\delta(x)$. This is subject to boundary conditions: (i) $u \to 0$ as $y \to \infty$, which implies $f' \to 0$ as $\xi \to \infty$; (ii) $f(\xi)$ is odd in ξ and so $f(0) = 0$; and (iii) the momentum flux is constant, $\int_{-\infty}^{\infty} f'^2 \, \mathrm{d}\xi = 1$. The solution is

$$f = c\tanh\left(\frac{1}{6}c\xi\right), \qquad \text{where} \qquad c^3 = 9/2. \tag{41}$$

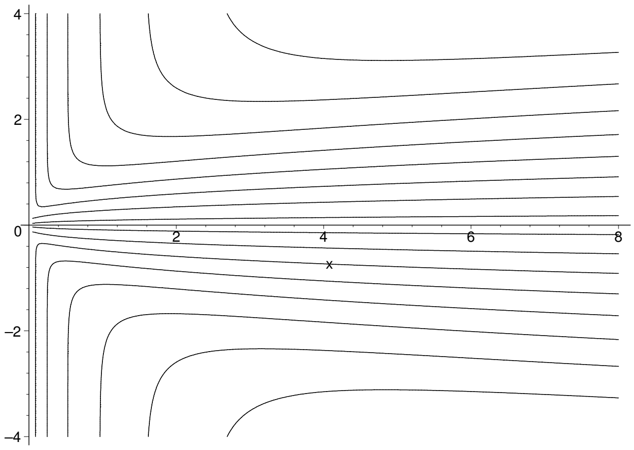


Figure 5: Streamlines of the jet with streamfunction $\psi = x^{1/3}\tanh(y/x^{2/3})$.

It is noteworthy that the mass flux associated with the jet motion increases with distance from source $\sim x^{1/3}$. This is because the jet 'entrains' surrounding fluid by the action of viscosity, causing it to flow along with it. Also the Reynolds number of the jet motion increases with distance and so eventually we expect the steady motion to breakdown into an unsteady turbulent flow.

5.7 Free surface boundary layers

At a free-surface we impose conditions of continuity of normal velocity and vanishing shear stress. Thus, in contrast the effect of a no-slip boundary condition, the velocity profile is only weakly modified as the surface is approached, due to the action of viscosity. This means that the dominant contribution to the dissipation occurs outside of the boundary layer.

A strategy to compute the dissipation at high Reynolds number is then to use the potential flow field solution, which is valid away from the boundary, and to compute the dissipation associated with it. Thus if the velocity field $\mathbf{u} = \nabla\phi$ with $\nabla^2\phi = 0$, the symmetric rate of strain tensor is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{42}$$

Thus an estimate of the viscous dissipation, $\mathcal{D}$ is

$$\mathcal{D} = 2\mu \int_V e_{ij}e_{ij}\ \mathrm{d}V = \mu \int_S n_i \frac{\partial}{\partial x_i} \left(\nabla\phi . \nabla\phi\right)\ \mathrm{d}S. \tag{43}$$

Example: Uniform flow past spherical bubble of radius a
The velocity potential is given by

$$\phi = \mathbf{U}.\,\mathbf{x}\left(1 + \frac{a^3}{2r^3}\right). \tag{44}$$

In spherical polar coordinates, we find that

$$|\mathbf{u}|^2 = |\nabla\phi|^2 = U^2\left(1 + \frac{a^3}{2r^3}\right)^2 - U^2r^2\cos^2\theta\left(\frac{3a^3}{r^5} - \frac{3a^6}{4r^8}\right). \tag{45}$$

Then on the surface of the bubble $\mathbf{n} = -\hat{\mathbf{r}}$ and so

$$\mathbf{n}.\,\nabla\left(|\nabla\phi|^2\right) = \frac{9}{2}\frac{U^2\sin^2\theta}{a}. \tag{46}$$

Integrating over the surface of the bubble gives $\mathcal{D} = 12\pi\mu U^2 a$. This is greater than the dissipation when the Reynolds number is vanishingly small ($\mathcal{D} = 4\pi\mu U^2 a$) - it had to be greater than this value by the Minimum Dissipation Theorem.
Example: Damping of deep water, linear waves
For a small amplitude free-surface oscillation of $\eta = \eta_0\sin(kx - \omega t)$ on water of infinite depth, the velocity potential is given by

$$\phi = -\frac{\eta_0 g}{\omega}\mathrm{e}^{kz}\cos(kx - \omega t), \tag{47}$$

where the dispersion relation gives $\omega^2 = gk$ and g denotes gravitational acceleration.

The speed of the motion is then given by

$$|\mathbf{u}|^2 = |\nabla\phi|^2 = \eta_0 gk\mathrm{e}^{2kz}. \tag{48}$$

The linearised position of the free surface is at $z = 0$ with normal vector $\mathbf{n} = \hat{\mathbf{z}}$ and so the dissipation per unit area is given by

$$\mathcal{D} = 2\mu\eta_0^2 gk^2. \tag{49}$$

To compute the damping to this wave, we suppose that η_0 is now a weak function of time and we evaluate the energy in the wave motion. The time-averaged potential energy is given by

$$PE = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^{\eta}\rho g z\ \mathrm{d}z\mathrm{d}t = \frac{1}{4}\rho g\eta_0^2. \tag{50}$$

The time-averaged kinetic energy is given by

$$KE = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_{-\infty}^{0}\frac{1}{2}\rho|\mathbf{u}|^2\ \mathrm{d}z\mathrm{d}t = \frac{1}{4}\rho g\eta_0^2. \tag{51}$$

The rate at which the total energy decreases is equal to the dissipation and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\rho g\eta_0^2\right) = -2\mu\eta_0^2 gk^2. \tag{52}$$

Treating the amplitude as time-varying, we thus find that

$$\eta_0(t) = \eta_0(0)\exp\left(-\frac{2\mu k^2 t}{\rho}\right). \tag{53}$$

For a wavelength of 30m in water with $\mu/\rho = 10^{-6}m^2s^{-1}$, this gives an e-folding time of 10^7s (approximately 0.3 years). Viscous dissipation is therefore only a very weak process in the oceans and this allows surface waves to be very long-lived.